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PREFACE.

In this little book I have attempted to put together consecutively, and in a small compass, all the essentials of elementary graphical work. It is intended mainly for that large class of students whose study of Algebra does not extend much beyond Quadratic Equations; at the same time it is believed that it will satisfy the requirements of most of the elementary Examinations which now include graphs.

My *Short Introduction to Graphical Algebra* has from the first maintained a high degree of popularity, but it is not sufficiently simple, and goes further than is necessary, for those whose algebraical reading is kept within somewhat narrow limits. Accordingly in the present case (although I have sometimes drawn upon the text and examples of my former work) I have pursued a different plan. I have made the explanations fuller in detail, while the easy examples for practice are more numerous and varied. In particular, I have endeavoured to give a fairly complete treatment of linear graphs before discussing functions of the second and higher degrees. Great stress has been laid upon the choice of suitable units, and in many examples the scale upon which the graphs should be drawn has been suggested. This should not only be helpful to students, but should also effect a great saving of time to teachers in correcting the work of their pupils.

If graphical work is to be really valuable to young students it must be convincing as well as interesting. In this connection it is most important that the squared paper used should be of

good quality and accurately ruled to inches and tenths of an inch. Experience shews that anything on a smaller scale (such as "millimetre paper") is practically worthless in the hands of beginners.

In the last section of the book the leading principles of linear graphs have been applied to some miscellaneous problems, illustrated by full-page diagrams. In the examples for practice which follow I have endeavoured to exclude problems which can be more readily solved by easy Arithmetic or Algebra, and to retain only those in which a graphical solution possesses real advantage and interest. The growing fashion of introducing graphs into all kinds of elementary work, where they are not wanted, and where they serve no useful purpose—either in illustration of guiding principles or in curtailing calculation—cannot be too strongly deprecated.

H. S. HALL.

April, 1905.

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EASY GRAPHS.

[*Arts.* 1–7 *may be read as soon as the student has had sufficient practice in substitutions involving negative quantities.* *Arts.* 8–21 *may be read after Simultaneous Equations of the first degree.* *Subsequent articles should be postponed until the student is acquainted with quadratic equations.*]

1. DEFINITION. Any expression which involves a variable quantity x, and whose value depends on that of x, is called a **function of x.**

Thus the expression $3x+8$ will have different values if different values are substituted for x, and is called a function of x of the first degree.

Similarly $2x^2+6x-7$, x^3-2x+1 are functions of x of the second and third degree respectively.

2. The words "function of x" are often briefly expressed by the symbol $f(x)$. If two quantities x and y are connected by a relation $y=f(x)$, by substituting a series of numerical values for x we can obtain a corresponding series of values for $f(x)$, that is for y.

Since in such a case the values of y *depend upon* the different values selected for x, it is sometimes convenient to call x the **independent variable**, and y the **dependent variable.**

3. Consider the function $x(9-x^2)$, and let its value be represented by y; so that $y=x(9-x^2)$.

Then, when $x=0$, $y=0\times 9=\ 0$,
" $x=1$, $y=1\times 8=\ 8$,
" $x=2$, $y=2\times 5=10$,
" $x=3$, $y=3\times 0=\ 0$,
" $x=4$, $y=4\times(-7)=-28$,

and so on.

By proceeding in this way we can find as many values of the function as we please. But we are often not so much concerned with the actual values which a function assumes for different values of the variable as with *the way in which the value of the function changes*. These variations can be very conveniently represented by a **graphical** method which we shall now explain.

4. Two straight lines XOX′, YOY′ are taken intersecting at right angles in O, thus dividing the plane of the paper into four spaces XOY, YOX′, X′OY′, Y′OX, which are known as the first, second, third, and fourth quadrants respectively.

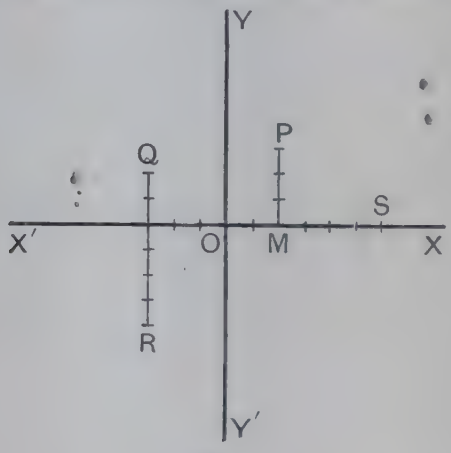

FIG. 1.

The lines X′OX, YOY′ are usually drawn horizontally and vertically; they are taken as lines of reference and are known as the **axis of x and y** respectively. The point O is called the **origin.** Values of x are measured from O along the axis of x, according to some convenient scale of measurement, and are called **abscissæ**, *positive* values being drawn to the *right* of O along OX, and *negative* values to the *left* of O along OX′.

Values of y are drawn parallel to the axis of y, from the ends of the corresponding abscissæ, and are called **ordinates.** These are *positive* when drawn *above* X′X, *negative* when drawn *below* X′X.

5. Suppose $y=3$, when $x=2$. To express this relation graphically we first mark off OM, 2 units in length, along OX; then at M we draw MP, 3 units in length, perpendicular to OX and above it. Thus the position of a point P is determined. Similarly any pair of corresponding values of x and y will determine a point relatively to the axes.

6. The abscissa and ordinate of a point taken together are known as its **coordinates.** A point whose coordinates are x and y is briefly spoken of as "the point (x, y)."

The process of marking the position of a point by means of its coordinates is known as **plotting the point.**

EXAMPLE. *Plot the points*

(i) $(-3, 2)$; (ii) $(-3, -4)$; (iii) $(6, 0)$.

(i) We proceed as in Art. 5, but since x is negative we first take 3 units to the *left* of O, that is along OX′; then 2 units at right angles to OX′ and above it. The resulting point Q is in the second quadrant. See Fig. 1.

(ii) Here we may briefly describe the process as follows: Take 3 steps to the *left*, then 4 *down*; the resulting point R is in the third quadrant.

(iii) Take 6 steps to the *right*, then *no steps either up or down* from OX. Thus the resulting point S is on the axis of X.

Note. The coordinates of the origin are (0, 0).

7. In practice it is convenient to use squared paper. Two intersecting lines should be chosen as axes, and slightly thickened to aid the eye, then one or more of the length-divisions may be taken as the linear unit.

We shall generally use paper ruled to tenths of an inch, but for greater clearness a larger scale will sometimes be adopted.

EXAMPLE 1. *Plot the points* $(5, 2)$, $(-3, 2)$, $(-3, -4)$, $(5, -4)$ *on squared paper. Find the area of the figure determined by these points, assuming the divisions on the paper to be tenths of an inch.*

Taking the points in the order given, it is easily seen that they are represented by P, Q, R, S in Fig. 2, and that they form a rectangle which contains 48 squares. Each of these is *one-hundredth* part of a *square* inch. Thus the area of the rectangle is 0·48 of a square inch.

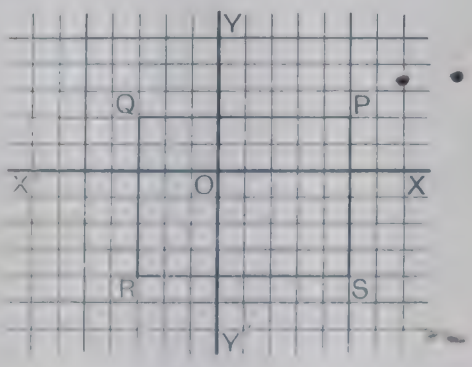

FIG. 2.

EXAMPLE 2. *The coordinates of the points* A *and* B *are* (7, 8) *and* (−5, 3): *plot the points and find the distance between them.*

After plotting the points as in the diagram, we may find AB approximately by direct measurement.

Or we may proceed thus:

Draw through B a line par^{l} to XX′ to meet the ordinate of A at C. Then ACB is a rt.-angled △ in which BC = 12, and AC = 5.

Now $AB^2 = BC^2 + AC^2$

$= 12^2 + 5^2$

$= 144 + 25$

$= 169.$

$\therefore AB = 13.$

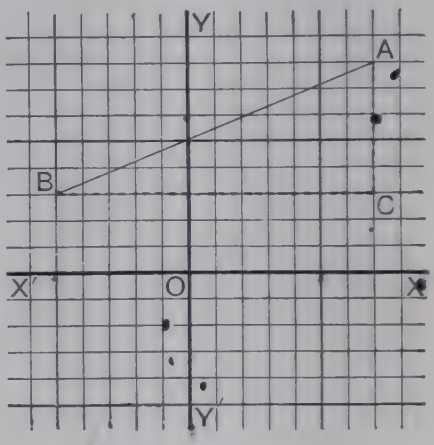

FIG. 3.

EXAMPLES I.

[*The following examples are intended to be done mainly by actual measurement on squared paper; where possible, they should also be verified by calculation.*]

Plot the following pairs of points and draw the line which joins them:

1. (3, 0), (0, 6).
2. (−2, 0), (0, −8).
3. (3, −8), (−2, 6).
4. (5, 5), (−2, −2).
5. (−2, 6), (1, −3).
6. (4, 5), (−1, 5).

7. Plot the points (3, 3), (−3, 3), (−3, −3), (3, −3), and find the number of squares contained by the figure determined by these points.

8. Plot the points (4, 0), (0, 4), (−4, 0), (0, −4), and find the number of units of area in the resulting figure.

9. Plot the points (0, 0), (0, 10), (5, 5), and find the number of units of area in the triangle.

10. Shew that the triangle whose vertices are (0, 0), (0, 6), (4, 3) contains 12 units of area. Shew also that the points (0, 0), (0, 6), (4, 8) determine a triangle of the same area.

11. Plot the points (5, 6), (−5, 6), (5, −6), (−5, −6). If each unit is supposed to represent one millimetre, find the area of the figure in square centimetres.

12. Plot the points (1, 3), (−3, −9), and shew that they lie on a line passing through the origin. Name the coordinates of other points on this line.

13. Plot the following points, and shew experimentally that each set lie in one straight line.

(i) (9, 7), (0, 0), (−9, −7); (ii) (−9, 7), (0, 0), (9, −7).

Explain these results theoretically.

14. Plot the following pairs of points; join the points in each case, and measure the coordinates of the mid-point of the joining line.

(i) (4, 3), (12, 7); (ii) (5, 4), (15, 16).

Shew *why* in each case the coordinates of the mid-point are respectively *half the sum of the abscissæ* and *half the sum of the ordinates* of the given points.

15. Plot the following pairs of points; and find the coordinates of the mid-point of their joining lines.

(i) (0, 0), (8, 10); (ii) (8, 0), (0, 10);

(iii) (0, 0), (−8, −10); (iv) (−8, 0), (0, −10).

16. Plot the following points, and calculate their distances from the origin.

(i) (15, 8); (ii) (−15, −8); (iii) (2·4″, ·7″); (iv) (−·7″, 2·4″).

Check your results by measurement.

17. Plot the following pairs of points, and in each case calculate the distance between them.

(i) (4, 0), (0, 3); (ii) (9, 8), (5, 5);

(iii) (15, 0), (0, 8); (iv) (10, 4), (−5, 12);

(v) (20, 8), (−15, 0); (vi) (20, 15), (−15, −3).

Verify your calculation by measurement.

18. Plot the eight points (0, 5), (3, 4), (5, 0), (4, −3), (−5, 0), (0, −5), (−4, 3), (−4, −3), and shew that they are all equidistant from the origin.

19. Plot the two following series of points :

(i) (5, 0), (5, 2), (5, 5), (5, −1), (5, −4) ;

(ii) (−4, 8), (−1, 8), (0, 8), (3, 8), (6, 8).

Shew that they lie on two lines respectively parallel to the axis of y, and the axis of x. Find the coordinates of the point in which they intersect.

20. Shew that the points (−3, 2), (3, 10), (7, 2) are the angular points of an isosceles triangle. Calculate and measure the lengths of the equal sides.

21. Explain by a diagram why the distances between the following pairs of points are all equal.

(i) $(a, 0), (0, b)$; (ii) $(b, 0), (0, a)$; (iii) $(0, 0), (a, b)$.

22. Draw the straight lines joining

(i) $(a, 0)$ and $(0, a)$; (ii) $(0, 0)$ and (a, a) ;

and prove that these lines bisect each other at right angles.

23. Find the perimeter of the triangle whose vertices are the points (7, 0), (0, 24), (−10, 0).

24. Draw the figure whose angular points are given by

(0, −3), (8, 3), (−4, 8), (−4, 3), (0, 0).

Find the lengths of its sides, taking the points in the above order.

25. Plot the points (13, 0), (0, −13), (12, 5), (−12, 5), (−13, 0), (−5, −12), (5, −12). Find their locus, (i) by measurement, (ii) by calculation.

26. Plot the points (2, 2), (−3, −3), (4, 4), (−5, −5), shewing that they all lie on a certain line through the origin. Conversely, shew that for *every* point on this line the abscissa and ordinate are equal.

27. If $y = 2x + 10$, find the values of y when x has the values 0, 1, 3, −2, −5. Plot the five points determined by these values, and shew experimentally that they lie on a straight line. Where does the line meet the axes ?

28. By giving different values to x find by trial a series of points whose coordinates satisfy the equation $2y = 5x$. Shew that they all lie on a straight line through the origin.

[*It will be convenient here to take two tenths of an inch as the unit.*]

Graph of a Function.

8. Let $f(x)$ represent a function of x, and let its value be denoted by y. If we give to x a series of numerical values we get a corresponding series of values for y. If these are set off as abscissæ and ordinates respectively, we plot a succession of points. If *all* such points were plotted we should arrive at a line, straight or curved, which is known as the **graph** of the *function* $f(x)$, or the **graph** of the *equation* $y=f(x)$. Thus the graph of the *function* $2x-5$ is the same as the graph of the *equation* $y=2x-5$.

The variation of the *function* for different values of the variable x is exhibited by the variation of the *ordinates* as we pass from point to point.

In practice a few points carefully plotted will usually enable us to draw the graph with sufficient accuracy.

9. The student who has worked intelligently through the preceding examples will have acquired for himself some useful preliminary notions which will be of service in the examples on simple graphs which we are about to give. In particular, before proceeding further he should satisfy himself with regard to the following statements:

(i) The coordinates of the origin are (0, 0).

(ii) For every point on the axis of x the value of y is 0.

Thus the graph of $y=0$ *is the axis of* x.

(iii) For every point on the axis of y the value of x is 0.

Thus the graph of $x=0$ *is the axis of* y.

(iv) The graph of all points which have the same abscissa is a line parallel to the axis of y.

Thus on page 6, Ex. 19, (i) gives a line parallel to the axis of y, and this line is the graph of $x=5$.

(v) The graph of all points which have the same ordinate is a line parallel to the axis of x.

Thus on page 6, Ex. 19, (ii) gives a line parallel to the axis of x, and this line is the graph of $y=8$.

(vi) The distance of any point $P(x, y)$ from the origin is given by $OP^2=x^2+y^2$. (See Ex. 18, p. 5.)

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EXAMPLE 1. *Plot the graph of* $y=x$.

When $x=0$, $y=0$; *thus the origin is one point on the graph.*

Also, when $x=1, 2, 3, \ldots -1, -2, -3, \ldots,$

$y=1, 2, 3, \ldots -1, -2, -3, \ldots.$

Thus the graph passes through O, and represents a series of points each of which has its ordinate equal to its abscissa, and is clearly represented by the straight line POP′ in Fig. 4.

EXAMPLE 2. *Plot the graph of* $y=x+3$.

Arrange the values of x and y as follows:

x	3	2	1	0	−1	−2	−3	...
y	6	5	4	3	2	1	0	...

By joining these points we obtain a line MN parallel to that in Example 1.

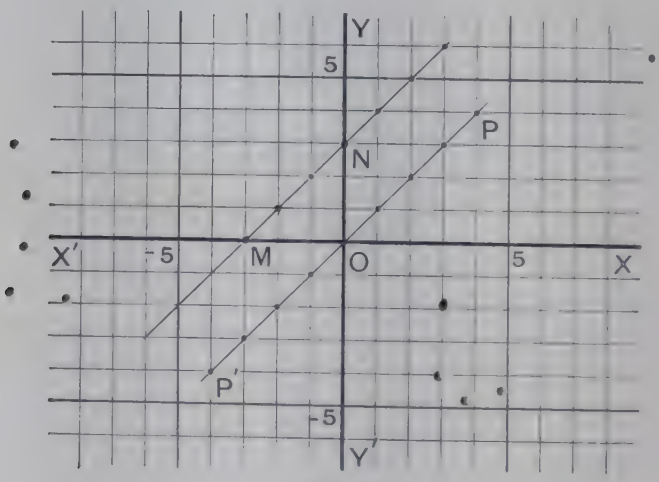

FIG. 4.

The results printed in larger and deeper type should be specially noted and compared with the graph. They shew that the distances ON, OM (usually called the *intercepts on the axes*) are obtained by separately putting $x=0$, $y=0$ in the equation of the graph.

Note. By observing that in Example 2 each ordinate is 3 units greater than the corresponding ordinate in Example 1, the graph of $y=x+3$ may be obtained from that of $y=x$ by simply producing each ordinate 3 units in the positive direction.

In like manner the equations

$$y=x+5, \quad y=x-5$$

represent two parallel lines on opposite sides of $y=x$ and equidistant from it, as the student may easily verify for himself.

EXAMPLE 3. *Plot the graphs represented by the equations:*

(i) $3y=2x$; (ii) $3y=2x+4$; (iii) $3y=2x-5$.

First put the equations in the equivalent forms:

(i) $y=\frac{2x}{3}$; (ii) $y=\frac{2x}{3}+\frac{4}{3}$; (iii) $y=\frac{2x}{3}-\frac{5}{3}$,

and in each case find values of y corresponding to

$$x=-3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3.$$

For example, in (i) we have the following values of y:

$$y=-2, \ -\tfrac{4}{3}, \ -\tfrac{2}{3}, \ 0, \ \tfrac{2}{3}, \ \tfrac{4}{3}, \ 2.$$

In plotting the corresponding points it will be found convenient to take *three* divisions of the paper as our unit.

The graphs will be found to be as in Fig. 5.

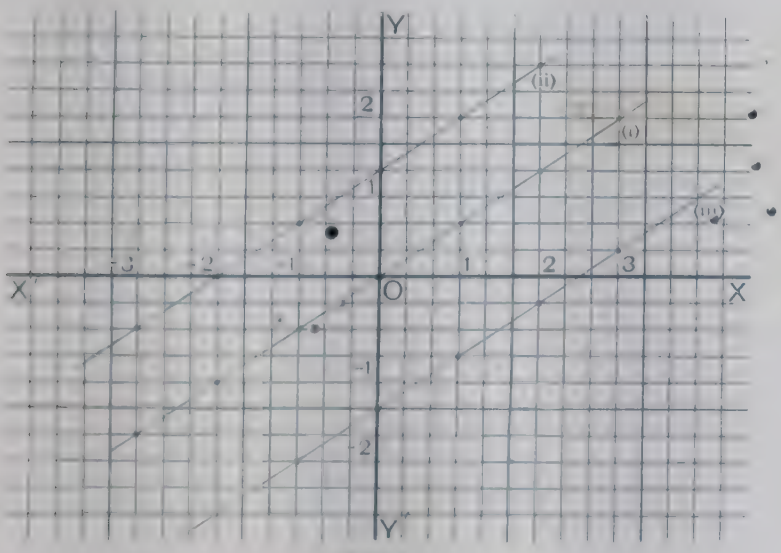

FIG. 5.

Each graph should be verified in detail by the student.

EXAMPLES II.

[*Examples* 1–18 *are arranged in groups of three; each group should be represented on the same diagram so as to exhibit clearly the position of the three graphs relatively to each other.*]

Plot the graphs represented by the following equations:

1. $y=5x$.	**2.** $y=5x-4$.	**3.** $y=5x+6$.
4. $y=-3x$.	**5.** $y=-3x+3$.	**6.** $y=-3x-2$.
7. $y+x=0$.	**8.** $y+x=8$.	**9.** $y+4=x$.
10. $4x=3y$.	**11.** $3y=4x+6$.	**12.** $4y+3x=8$.
13. $x-5=0$.	**14.** $y-6=0$.	**15.** $5y=6x$.
16. $3x+4y=10$.	**17.** $4x+y=9$.	**18.** $5x-2y=8$.

19. Shew by careful drawing that the three last graphs have a common point whose coordinates are 2, 1.

20. Shew by careful drawing that the equations

$$x+y=10, \quad y=x-4$$

represent two straight lines at right angles.

21. Draw on the same axes the graphs of $x=5$, $x=9$, $y=3$, $y=11$. Find the number of units of area enclosed by these lines.

22. Taking one-tenth of an inch as the unit of length, find the area included between the graphs of $x=7$, $x=-3$, $y=-2$, $y=8$.

23. Find the area included by the graphs of

$$y=x+6, \quad y=x-6, \quad y=-x+6, \quad y=-x-6.$$

24. With one millimetre as linear unit, find in square centimetres the area of the figure enclosed by the graphs of

$$y=2x+8, \quad y=2x-8, \quad y=-2x+8, \quad y=-2x-8.$$

25. Draw the graphs of the following equations:

$$x+y=5, \quad 2x-y=10, \quad 2x+3y=-30, \quad 3y-x=15.$$

If the paper is ruled to tenths of an inch, shew that the graphs include an area of 1·5 sq. in.

10. The student should now be prepared for the following statements :

(i) For all numerical values of a the equation $y=ax$ represents a straight line through the origin.

If a is positive x and y have the same sign, and the line lies in the first and third quadrants ; if a is negative x and y have opposite signs, and the line lies in the second and fourth quadrant.

In either case a is called the **slope** of the line.

(ii) For all numerical values of a and b the equation $y=ax+b$ represents a line parallel to $y=ax$, and cutting off an intercept b from the axis of y.

The graph of $y=ax+b$ is fixed in position as long as a and b retain the same values.

If a alone is altered, the line will have a different direction but will still cut the axis of y at the same distance (b) from the origin.

If b alone is altered, the line will still be parallel to $y=ax$, but will cut the axis of y at a different distance from the origin, further or nearer according as b is greater or less.

Since the values a and b fix the position of the line we are considering in any one piece of work, they are called the **constants** of the equation.

Note. The *slope* of $y=ax+b$ is the same as that of $y=ax$.

(iii) From the way in which the plotted points are determined from an equation, it follows that the graph passes through all points whose coordinates satisfy the equation, and through no other points.

11. Since every equation involving x and y only in the first degree can be reduced to one of the forms $y=ax$, $y=ax+b$, it follows that *every simple equation connecting two variables represents a straight line*. For this reason an expression of the form $ax+b$ is said to be a **linear function** of x, and an equation such as $y=ax+b$, or $ax+by+c=0$, is said to be a **linear equation.**

EXAMPLE. *Shew that the points* (3, −4), (9, 4), (12, 8) *lie on a straight line, and find its equation.*

Assume $y=ax+b$ as the equation of the line. If it passes through the first two points given, their coordinates must satisfy this equation.

Substituting $x=3$, $y=-4$, we have

$$-4=3a+b. \quad \text{.....(i)}$$

Again substituting $x=9$, $y=4$, we have

$$4=9a+b. \quad \text{.....(ii)}$$

By solving equations (i) and (ii) we obtain

$$a=\frac{4}{3},\ b=-8.$$

Hence $$y=\frac{4}{3}x-8, \text{ or } 4x-3y=24,$$

is the equation of the line passing through the first two points. Since $x=12$, $y=8$ satisfies this equation, the line also passes through (12, 8). This example may be verified graphically by plotting the line which joins *any two* of the points and shewing that it passes through the third.

12. Since a straight line can always be drawn when *any* two points on it are known, in drawing a *linear* graph only two points need be plotted. The points where the line meets the axes can be readily found by putting $y=0$, $x=0$, successively in the equation, and these two points will always suffice, though they are not always the best to select.

EXAMPLE. *Draw the graph of* $4x-3y=13$.

If we find the intercepts on the axes we have

$$\text{when } y=0,\ x=\frac{13}{4} \text{ (intercept on the } x\text{-axis)},$$

$$\text{and when } x=0,\ y=-\frac{13}{3} \text{ (intercept on the } y\text{-axis)}.$$

As both of these values involve fractions of the unit, it would be difficult to draw the line with sufficient accuracy.

In such a case it is better to find by trial *integral* values of x and y which satisfy the equation.

Thus when $x=1$, $y=-3$, and when $y=1$, $x=4$.

The graph can now be drawn by joining the points (1, −3), (4, 1).

Application to Simultaneous Equations.

13. When there is only one simple equation connecting x and y, it is possible to find as many pairs of values of x and y as we please which satisfy the given equation. We now see that this is equivalent to saying that we may find as many points as we please on any given straight line. If, however, we have two *simultaneous* equations between x and y, there can only be one pair of values which will satisfy both equations. This is equivalent to saying that two straight lines can have only one common point.

EXAMPLE. *Solve graphically the equations:*

$$\text{(i) } 3y - x = 6, \quad \text{(ii) } 3x + 5y = 38.$$

In (i) the intercepts on the axes are -6, 2. Thus the line is found by joining $\mathrm{P}(-6, 0)$ and $\mathrm{P}'(0, 2)$.

In (ii) when $x=1$, $y=7$, and when $y=1$, $x=11$.

Thus the line is found by joining $\mathrm{Q}(1, 7)$ and $\mathrm{Q}'(11, 1)$.

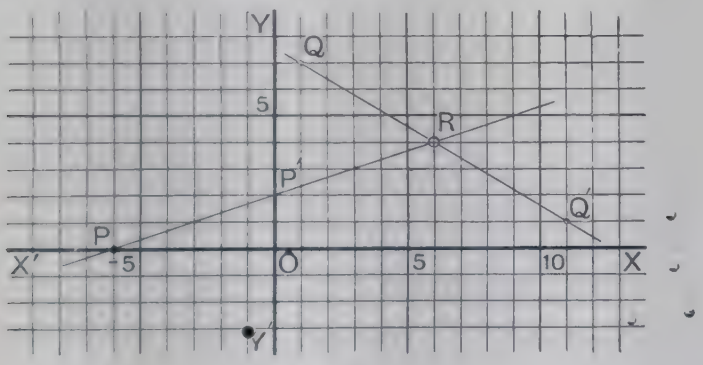

FIG. 6.

It is seen from the diagram that these lines intersect at the point R whose coordinates are 6, 4. Thus the solution of the given equations is $x=6$, $y=4$.

The student should verify this result by solving the equations algebraically by any of the methods applicable to simultaneous equations.

14. It will now be seen that the process of solving two linear simultaneous equations is equivalent to finding the coordinates of the point at which their graphs meet.

EXAMPLE. *Draw the graphs of*

(i) $5x+6y=60$, (ii) $6y-x=24$, (iii) $2x-y=7$;

and shew that they represent three lines which meet in a point.

In (i) when $y=0$, $x=12$; when $x=0$, $y=10$.

Thus the intercepts on the axes are 12 and 10, and the graph is the line PP′.

In (ii) when $x=0$, $y=4$; when $x=12$, $y=6$, and the graph is the line joining Q (0, 4) to Q′ (12, 6).

In (iii) when $x=0$, $y=-7$; when $x=8$, $y=9$, and the graph is the line joining R (0, −7) to R′ (8, 9).

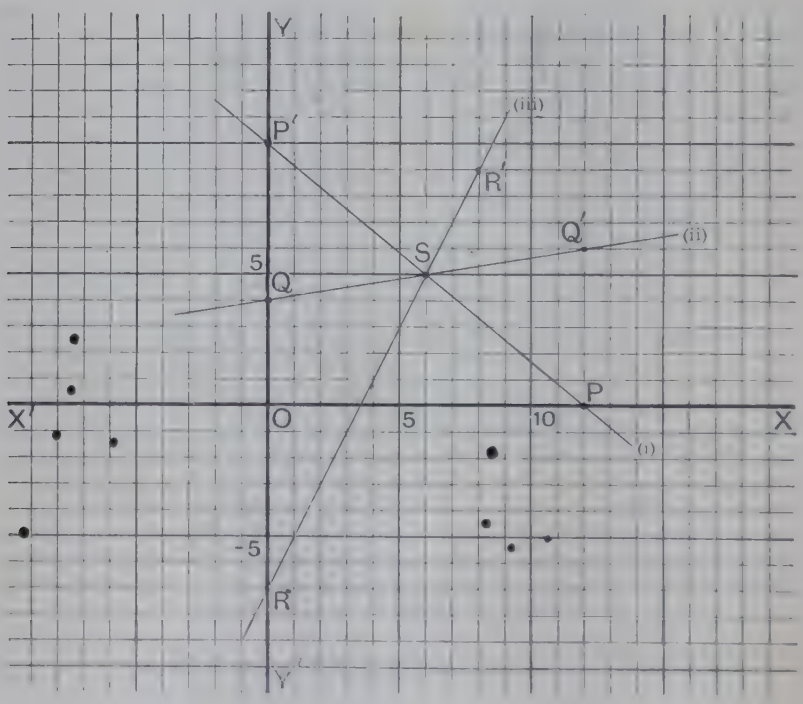

FIG. 7.

From the diagram it is evident that these three lines all pass through the point S whose coordinates are 6, 5.

15. Two simultaneous equations lead to no finite solution if they are inconsistent with each other. For example, the equations

$$x+3y=2, \quad 3x+9y=8$$

are inconsistent, for the second equation can be written $x+3y=2\frac{2}{3}$, which is clearly inconsistent with $x+3y=2$. The graphs of these two equations will be found to be two parallel straight lines which have no finite point of intersection.

Again, two simultaneous equations must be independent. The equations

$$4x+3y=1, \quad 16x+12y=4$$

are not independent, for the second can be deduced from the first by dividing throughout by 4. Thus *any pair of values* which will satisfy one equation will satisfy the other. Graphically these two equations represent two coincident straight lines which of course have an unlimited number of common points.

EXAMPLES III.

Solve the following equations graphically:

1. $y=2x+3$, $y+x=6$.

2. $y=3x+4$, $y=x+8$.

3. $y=4x$, $2x+y=18$.

4. $2x-y=8$, $4x+3y=6$.

5. $3x+2y=16$, $5x-3y=14$.

6. $6y-5x=18$, $4x=3y$.

7. $2x+y=0$, $y=\frac{4}{3}(x+5)$.

8. $2x-y=3$, $3x-5y=15$.

9. $2y=5x+15$, $3y-4x=12$.

10. Shew that the straight lines given by the equations

$$9y=5x+65, \quad 5x+2y+10=0, \quad x+3y=11,$$

meet in a point. Find its coordinates.

11. Prove by graphical representation that the three points (3, 0), (2, 7), (4, −7) lie on a straight line. Where does this line cut the axis of y?

12. Prove that the three points (1, 1), (−3, 4), (5, −2) lie on a straight line. Find its equation. Draw the graph of this equation, shewing that it passes through the given points.

13. Shew that the three points (3, 2), (8, 8), (−2, 4) lie on a straight line. Prove algebraically and graphically that it cuts the axis of x at a distance $1\frac{1}{3}$ from the origin.

16. Measurement on Different Scales. For the sake of simplicity we have hitherto measured abscissæ and ordinates on the same scale, but there is no necessity for so doing, and it will often be convenient to measure the variables on different scales suggested by the particular conditions of the question.

For example, in drawing the graph of $y=11x+6$,

when x has the values $-2, -1, 0, 1, 2, 3,$

the corresponding values of y are $-16, -5, 6, 17, 28, 39.$

Thus some of the ordinates are much larger than the abscissæ, and rapidly increase as x increases.

On plotting these points with x and y measured on the same scale, it will be found that with a small unit the graph is inconveniently placed with regard to the axes. If a larger unit is employed the graph requires a diagram of inconvenient size.

[The student should prove this for himself experimentally.]

The inconvenience can be obviated by measuring the values of y on a considerably smaller scale than those of x.

For example, let us take $\frac{1}{20}$ of an inch as unit for y and one inch as unit for x; then the graph of $y=11x+6$ will be as in Fig. 8, in which the line has been drawn by joining the points (0, 6), (2, 28).

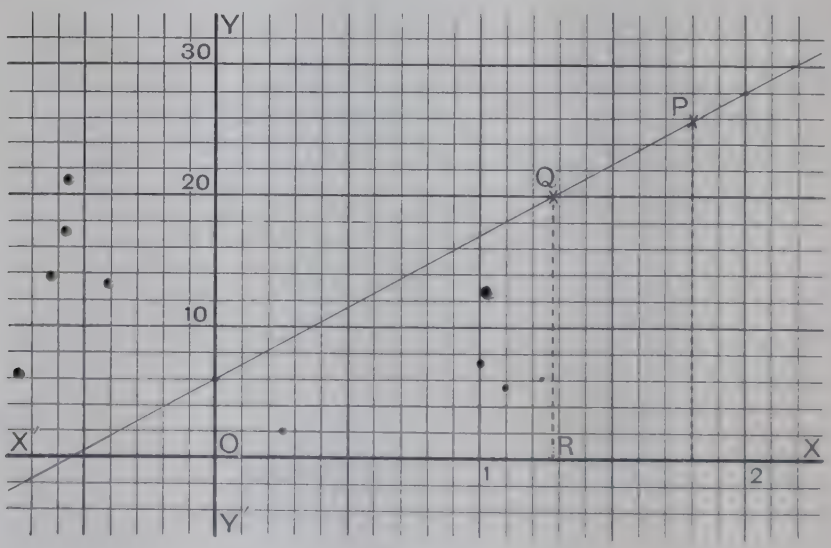

Fig. 8.

Speaking generally, whenever one variable increases much more rapidly than the other, a small unit should be chosen for the rapidly increasing variable and a large one for the other.

17. When a graph has been accurately drawn from plotted points, it can be used to *read off* (without calculation) corresponding values of the variables at intermediate points. Or if one coordinate of a point on the graph is known the other can be found by measurement. Sometimes, of course, results so obtained will only be approximate, but, as will be seen later, some of the most valuable results of graphical methods are arrived at in this way. The process is known as **interpolation.**

EXAMPLE. *From the graph of the expression* $11x+6$, *find its value when* $x=1{\cdot}8$. *Also find the value of* x *which will make the expression equal to* 20.

Put $y=11x+6$, then the graph is that given in Fig. 8. Now we see that $x=1{\cdot}8$ at the point P, and here $y=26$, nearly.

Again $y=20$ at the point Q; and $x=\mathrm{OR}=1{\cdot}28$, approximately. In obtaining this last result we observe that OR is greater than 1·2 and less than 1·3, and we mentally divide the tenth in which R falls into *ten* equal parts (*i.e.* into *hundredths of the unit*) and judge as nearly as possible how many of these hundredths are to be added to 1·2.

EXAMPLES IV.

[*In some of the following Examples the scales are specified; in others the student is left to select suitable units for himself. When two or more equations are involved in the same piece of work, their graphs must all be drawn on the same scale. In every case the units employed should be marked on the axes.*]

1. By finding the intercepts on the axes draw the graphs of

(i) $15x+20y=6$; (ii) $12x+21y=14$.

In (i) take 1 inch for unit, and in (ii) take six tenths of an inch as unit. In each case explain why the unit is convenient.

2. Solve $y=10x+8$, $7x+y=25$ graphically.

[Unit for x, one inch; for y, one tenth of an inch.]

3. With the same units as in Ex. 2 draw the graph of the function $\frac{36-5x}{3}$. From the graph find the value of the function when $x=1{\cdot}8$; also find for what value of x the function becomes equal to 8.

4. On one diagram draw the graphs of

$$y=5x+11, \qquad 10x-2y=15.$$

What is the slope of these graphs? Find the length of the y-axis intercepted between them.

5. Draw the graphs of the equations :

$$3{\cdot}4x+5y=17, \quad x-y=0{\cdot}8, \quad y-0{\cdot}5x=0{\cdot}45 ;$$

and shew that they all pass through one point.

6. Draw the triangle whose sides are represented by the equations : $3y-x=9, \quad x+7y=11, \quad 3x+y=13 ;$

and find the coordinates of the vertices.

7. With an inch as unit draw the triangle whose sides are given by the following equations, and find its area.

$$10y+2x=31, \quad y=3{\cdot}5x, \quad 5y-2x=6{\cdot}5.$$

8. I want a ready way of finding approximately 0·866 of any number up to 10. Justify the following construction. Join the origin to a point P whose coordinates are 10 and 8·66 (1 inch being taken as unit) ; then the ordinate of any point on OP is 0·866 of the corresponding abscissa. Read off from the diagram,

$$0{\cdot}866 \text{ of } 3, \ 0{\cdot}866 \text{ of } 6{\cdot}5, \ 0{\cdot}866 \text{ of } 4{\cdot}8, \text{ and } \frac{1}{0{\cdot}866} \text{ of } 5.$$

18. The last example gives a simple illustration of a graph used as a "ready reckoner." We shall now work two other examples of this kind.

EXAMPLE 1. *Given that* 5·5 *kilograms are roughly equal to* 12·125 *pounds, shew graphically how to express any number of pounds in kilograms. Express* $7\frac{1}{2}$ *lbs. in kilograms, and* $4\frac{1}{4}$ *kilograms in pounds.*

Let y kilograms be equal to x pounds, then evidently we have $y=\frac{5{\cdot}5}{12{\cdot}125}x$, which is the equation of a straight line *through the origin.* Hence measuring pounds horizontally and kilograms vertically the required graph is obtained at once by joining the origin to the point whose coordinates are 12·125 and 5·5.

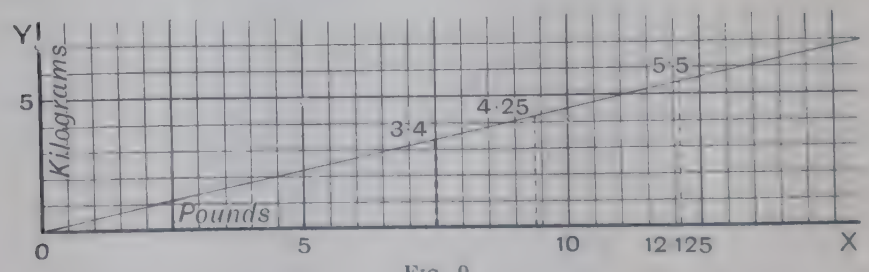

FIG. 9.

By measurement it will be found that $7\frac{1}{2}$ lbs. = 3·4 kilograms, and $4\frac{1}{4}$ kilograms = 9·37 lbs.

[The graph should be drawn by the student on a larger scale.]

EXAMPLE 2. *The expenses of a school are partly constant and partly proportional to the number of boys. The expenses were* £650 *for* 105 *boys, and* £742 *for* 128. *Draw a graph to represent the expenses for any number of boys; find the expenses for* 115 *boys, and the number of boys that can be maintained at a cost of* £710.

If the total expenses for x boys are represented by £y, the variable part may be denoted by £ax, and the constant part by £b. Hence x and y satisfy a linear equation $y=ax+b$, where a and b are constant quantities. Hence the graph is a straight line.

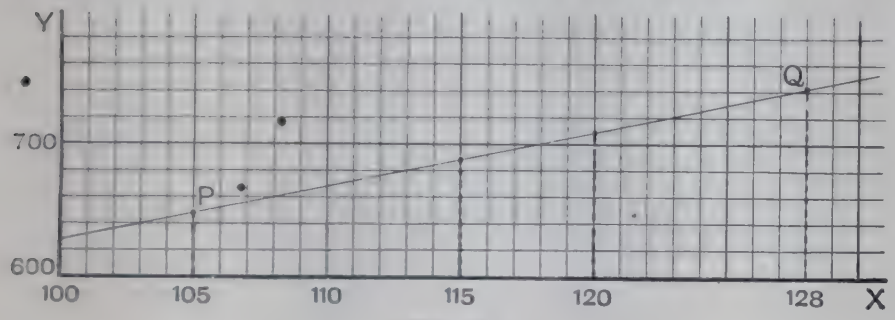

FIG. 10.

As the numbers are large, it will be convenient if we begin measuring ordinates at 600, and abscissæ at 100. This enables us to bring the requisite portion of the graph into a smaller compass. When $x=105$, $y=650$; and when $x=128$, $y=742$. Thus two points P and Q are found, and the line PQ is the required graph.

By measurement we find that when $x=115$, $y=690$; and that when $y=710$, $x=120$. Thus the required answers are £690, and 120 boys.

EXAMPLES IV. (*Continued.*)

9. Given that 6·01 yards = 5·5 metres, draw the graph shewing the equivalent of any number of yards when expressed in metres.

Shew that 22·2 yards = 20·3 metres, approximately.

10. Draw a graph shewing the relation between equal weights in grains and grams, having given that 10·8 grains = 1·17 grams.

Express (i) 3·5 grams in grains.

(ii) 3·09 grains as a decimal of a gram.

11. If 3·26 in. are equivalent to 8·28 cm., shew how to find graphically the number of inches corresponding to a given number of centimetres. Obtain the number of inches in a metre, and the number of centimetres in a yard. Find the equation to the graph.

12. The highest marks gained in an examination were 136, and these are to be raised so that the maximum is 200. Shew how this may be done by means of a graph, and read off, to the nearest integer, the final marks of candidates who scored 61 and 49 respectively.

13. A man buys 100 eggs for 5s. and has to pay 1s. 8d. for carriage. He wishes to sell them so as to gain 15 per cent. on his whole outlay. Draw a graph to shew to the nearest penny the selling price of any number of eggs up to 100, and read off the price of 65. From the graph find the number of eggs which could be bought for 6s. 8d.

14. The highest and lowest marks gained in an examination are 297 and 132 respectively. These have to be reduced in such a way that the maximum for the paper (200) shall be given to the first candidate, and that there shall be a range of 150 marks between the first and last. Draw a graph from which the reduced marks may be read off, and find what marks should be given to candidates who gained 200, 262, 163 marks in the examination.

Find the equation between x, the actual marks gained, and y, the corresponding marks when reduced.

15. For a certain book it costs a publisher £100 to prepare the type and 2s. to print each copy. Find an expression for the total cost in pounds of x copies. Make a diagram on a scale of 1 inch to 1000 copies, and 1 inch to £100 to shew the total cost of any number of copies up to 5000. Read off the cost of 2500 copies, and the number of copies costing £525.

19. In all the cases at present considered the graph has been a straight line obtained by first selecting values of x and y which satisfy *an equation of the first degree*, and then drawing a line so as to pass through the plotted points. The method is quite general, and it is easy to see that it may be applied when the variables are connected by an equation *which is not linear*. In such a case it will be found that a line drawn through the plotted points will take the form of some *curve* differing in shape according to the equation which connects the variables. Before discussing such cases we may observe that, whenever two variable quantities depend on each other so that a change in one produces a corresponding change in the other, we can draw a graph to exhibit their variations without knowing any algebraical relation between them, *provided that we are furnished with a sufficient number of corresponding values accurately determined.*

But we frequently have to deal with cases in which a limited number of corresponding values of two variables have been obtained by observation or experiment. In such cases the data may involve inaccuracies, and consequently the position of the plotted points cannot be absolutely relied on. Moreover we cannot correct irregularities in the graph by plotting other points selected at discretion. One method of procedure is to join successive points by *straight* lines. The graph will then be represented by an irregular broken line, sometimes with abrupt changes of direction as we pass from point to point. In cases where no great accuracy of detail is required this simple method is often used to illustrate statistical results. A familiar instance is a Weather Chart giving the height of the barometer at equal intervals of time.

The chief disadvantage of this method is that, although it gives a general idea of the total change that has taken place between the plotted points, it furnishes no accurate information with regard to intermediate points.

EXAMPLE. *The readings of a thermometer taken at intervals of 2 hours beginning at* 10 a.m. *were* 62·5°, 64°, 69·6°, 69°, 66·5°, 65·7°.

Draw a chart to shew the changes of temperature.

Measuring degrees vertically and hours horizontally, with the scales indicated on the diagram, we obtain the broken line PQRSTV shewn in Fig. 11.

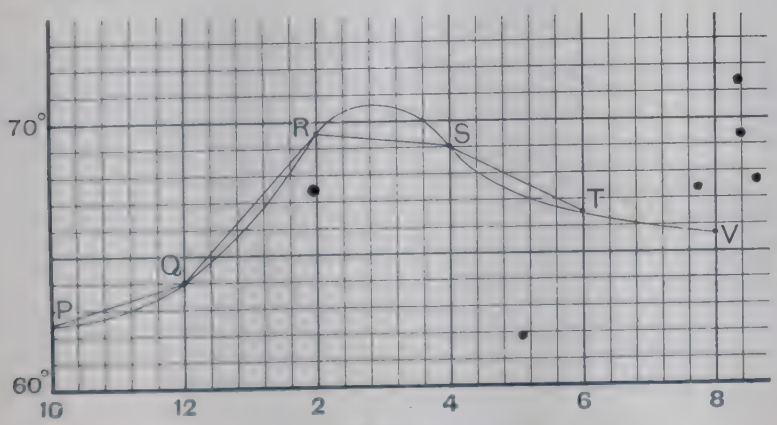

FIG. 11.

But it is contrary to experience to suppose that the abrupt changes of direction at Q and R accurately represent the change of temperature at noon and 2 p.m. respectively. Moreover, it is probable that the maximum temperature occurred at some time between 2 and 4, and not at the time represented at R, the highest of the

plotted points. Now if the chart had been obtained by means of a self-registering instrument, the graph (representing change from instant to instant instead of at long intervals) would probably have been somewhat like the continuous waving curve drawn through the points previously registered. From this it would appear that the maximum temperature occurred shortly before 3 p.m., and that TV (which represents a very gradual change) is the only portion of the broken line which records with any degree of accuracy the variation in temperature during two consecutive hours.

20. Although in the last example we were able to indicate the form of the curved line which from the nature of the case *seemed most probable*, it is evident that any number of curves can be drawn through a limited number of plotted points. In such a case the best plan is to draw a curve to lie as evenly as possible among the plotted points, passing through some perhaps, and with the rest fairly distributed on either side of the curve. As an aid to drawing an even continuous curve (usually called a *smooth* curve), a thin piece of wood or other flexible material may be bent into the requisite shape, and held in position while the line is drawn. A contrivance known as "Brooks' Flexible Curve" will often be found useful. When the plotted points lie approximately on a straight line, the simplest plan is to use a piece of tracing paper or celluloid on which a straight line has been drawn. When this has been placed in the right position the extremities can be marked on the squared paper, and by joining these points the approximate graph is obtained.

When the graph is linear it can be produced to any extent within the limits of the paper and so any value of one of the variables being determined, the corresponding value of the other can be read off. When large values are in question this method is inconvenient ; the following Example illustrates the method of procedure in such cases.

EXAMPLE. *Corresponding values of* x *and* y, *some of which are slightly inaccurate, are given in the following table* :

x	1	4	6·8	8	9·5	12	14·4
y	4	8	12·2	13	14·8	20	24·8

Draw the most probable graph and find its equation. Also find the value of y *corresponding to* $x = 80$.

Let 1 inch be taken to represent 5 units along OX, and 20 units along OY.

After carefully plotting the given points we see that a straight line can be drawn passing through three of them and lying evenly among the others. This is the required graph.

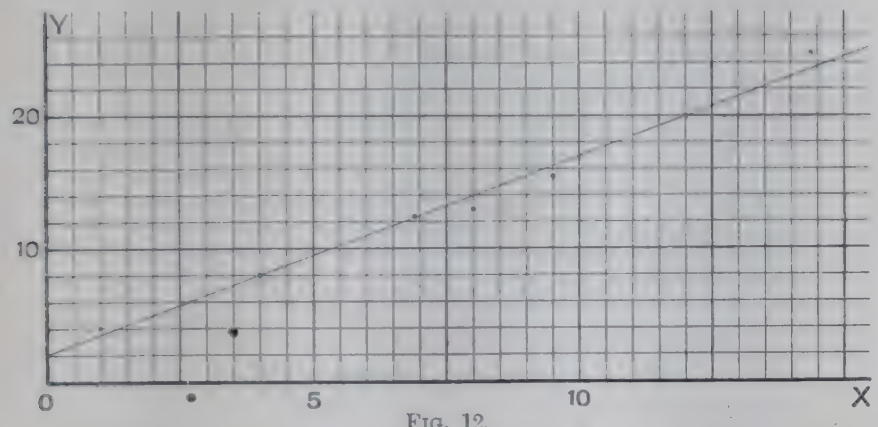

FIG. 12.

Assuming $y = ax + b$ for its equation, we can find the values of a and b by substituting the coordinates of two points through which the line passes.

Thus putting $x=4$, $y=8$, we obtain $8=4a+b$;
again, when $x=12$, $y=20$, we have $20=12a+b$.

By solving these equations we obtain $a=1{\cdot}5$, $b=2$.

Hence the equation of the graph is $y=1{\cdot}5x+2$, and the coordinates of any number of points on the line may now be found by trial.

Thus when $x=80$, $y=122$.

21. We shall now give an Example to illustrate a method common in the laboratory or workshop, the object being to determine the law connecting two variables when certain simultaneous values have been found by experiment or observation.

EXAMPLE. *In a certain machine P is the force in pounds required to raise a weight of W pounds. The following corresponding values of P and W were obtained experimentally:*

P	2·48	3·9	6·8	8·8	9·2	11	13·3
W	21	36·25	66·2	87·5	103·75	120	152·5

By plotting these values on squared paper draw the graph connecting P and W, and read off the value of P when $W=70$. Also determine a linear law connecting P and W; find the force necessary to raise a weight of 310 lbs., and also the weight which could be raised by a force of 180·6 lbs.

As the page is too small to exhibit the graphical work on a convenient scale we shall merely indicate the steps of the solution, which is similar in detail to that of the last example.

Plot the values of P vertically and the values of W horizontally. Taking 0·5 of an inch as unit for P, and 0·1 of an inch as unit for W, it will be found that a straight line can be drawn through the points corresponding to the results marked with an asterisk, and lying evenly among the other points. From this graph we find that when $W=70$, $P=7$.

Assume $P=aW+b$, and substitute for P and W from the values corresponding to the two points through which the line passes. By solving the resulting equations we obtain $a=0{\cdot}08$, $b=1{\cdot}4$. Thus the linear equation connecting P and W is $P=0{\cdot}08\,W+1{\cdot}4$.

This is called the **Law of the Machine.**

From this equation, when $W=310$, $P=26{\cdot}2$; and when $P=180{\cdot}6$, $W=2240$.

Thus a force of 26·2 lbs. will raise a weight of 310 lbs.; and when a force of 180·6 lbs. is applied the weight raised is 2240 lbs. or 1 ton.

[The student should verify all the details of the work for himself.]

Note. The equation of the graph is not only useful for determining results difficult to obtain graphically, but it can always be used to check results found by measurement.

EXAMPLE 2. *The following table gives statistics of the population of a certain country, where P is the number of millions at the beginning of each of the years specified.*

Year	1830	1835	1840	1850	1860	1865	1870	1880
P	20	22·1	23·5	29·0	34·2	38·2	41·0	49·4

Let t *be the time in years from* 1830. *Plot the values of P vertically and those of* t *horizontally and exhibit the relation between P and* t *by a simple curve passing fairly evenly among the plotted points. Find what the population was at the beginning of the years* 1848 *and* 1875.

Take 0·1 of an inch as unit in each case; also it will be convenient if we begin measuring abscissæ at 1830, and ordinates at 20.

The graph is given in Fig. 13 on the opposite page: it will be seen that it passes exactly through the extreme points and lies evenly among the others.

The populations in 1848 and 1875, at the points A and B respectively, will be found to be 27·8 millions and 45·3 millions.

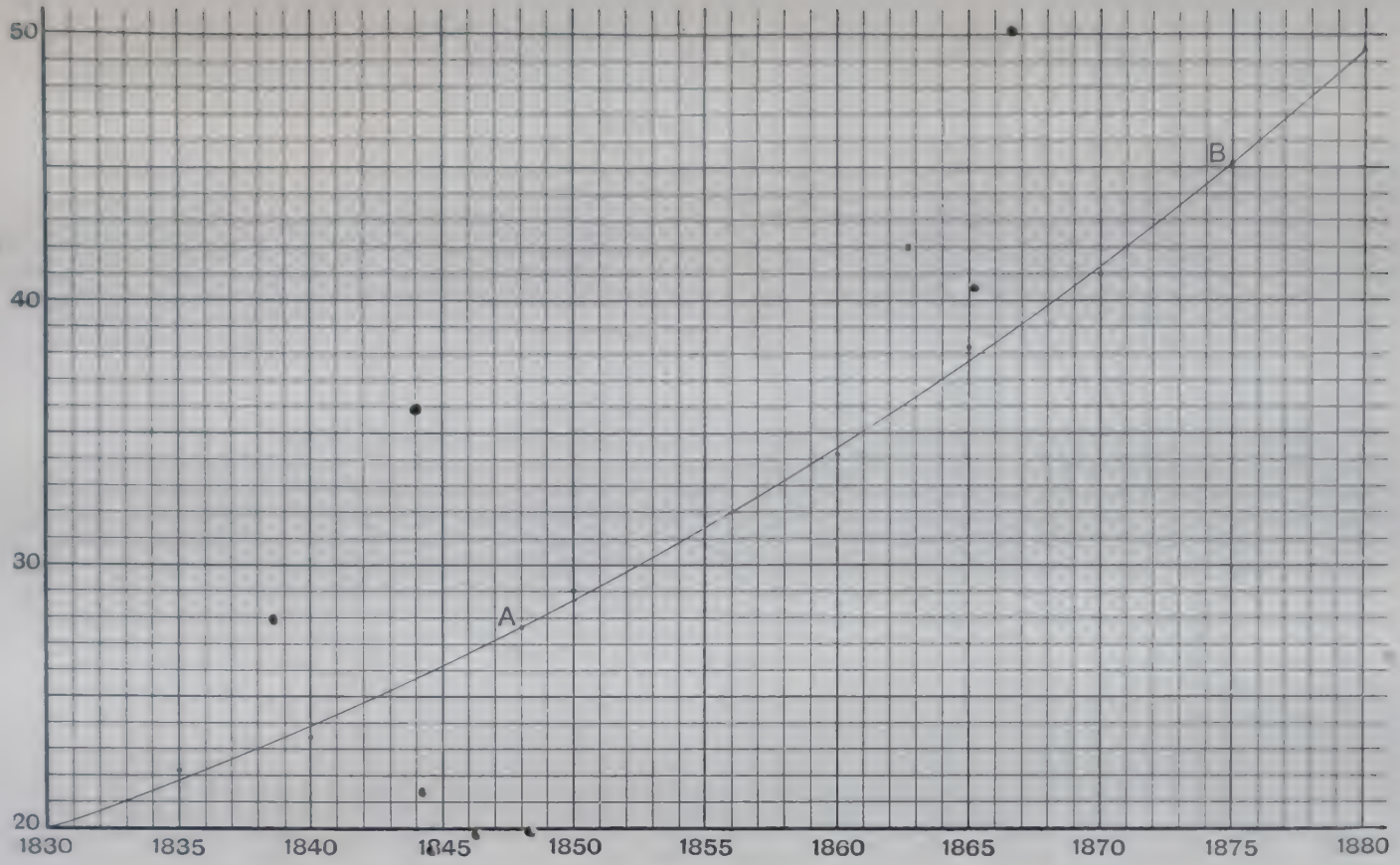
50
40
30
20
1830
1835
1840
1845
1850
1855
1860
1865
1870
1875
1880
A
B

FIG. 13.

EXAMPLES V.

[*In Examples 1–4 the plotted points may be joined by straight lines. In other cases the graph is to be a straight line or smooth curve lying evenly among the plotted points.*]

1. In a term of 11 weeks a boy's places in his Form were as follows:

8, 6, 11, 10, 9, 6, 6, 4, 2, 1, 1.

Shew these results by means of a graph.

2. The mean heights of the barometer in inches for the first 10 days of January 1904, recorded at the Royal Observatory, were as follows:

29·21, 29·12, 29·00, 29·25, 29·37, 29·26, 29·46, 28·83, 28·66, 28·76.

Exhibit these variations by means of a chart.

3. The highest and lowest prices of Consols for the years 1895 to 1904 were as follows:

Year	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Highest	$108\frac{1}{8}$	$113\frac{7}{8}$	$113\frac{7}{8}$	$113\frac{7}{8}$	$111\frac{1}{2}$	$103\frac{1}{4}$	$97\frac{3}{8}$	$97\frac{7}{8}$	$93\frac{3}{8}$	$91\frac{1}{4}$
Lowest	$103\frac{1}{2}$	$105\frac{1}{4}$	$110\frac{3}{8}$	$106\frac{3}{4}$	$97\frac{3}{4}$	$96\frac{3}{4}$	91	$92\frac{5}{16}$	$86\frac{7}{8}$	85

Make a chart to shew these variations graphically on the same diagram.

[A convenient scale will be: one inch to £10 vertically, beginning at 85, and 0·5 of an inch to 1 year horizontally.]

4. Make a chart to shew the variations in French Imports and Exports (in millions of pounds), for the years 1896 to 1903 inclusive, from the following data:

Imports	50·1	53·3	51·3	53·0	53·6	51·2	50·6	49·9
Exports	20·6	19·5	20·5	22·2	25·8	23·7	22·2	23·1

5. Corresponding values of x and y are given in the following table:

x	3	6·5	12	14	21	28·6	31·5
y	4	4·8	6·7	7	8·5	11	11·5

Draw the most probable graph, and find its equation. Find the value of x when $y=11·5$, and the value of y when $x=10$.

6. Plot on squared paper the following measured values of x and y, and determine the most probable equation between x and y:

x	3	5	8·3	11	13	15·5	18·6	23	28
y	2	2·2	3·4	3·8	4	4·6	5·4	6·2	7·25

7. Corresponding values of x and y are given in the following table:

x	1	3·1	6	9·5	12·5	16	19	23
y	2	2·8	4·2	5·3	6·6	8·3	9	10·8

Supposing these values to involve errors of observation, draw the graph approximately, and determine the most probable equation between x and y. Find the correct value of y when $x=19$, and the correct value of x when $y=2·8$.

8. At different ages the mean after-lifetime ("expectation of life") of males, calculated on the death rates of 1871–1880, was given by the following table:

Age	6	10	14	18	22	26	27
Expectation	50·38	47·60	44·26	40·96	37·89	34·96	34·24

Draw a graph to shew the expectation of any male between the ages of 6 and 27, and from it determine the expectation of persons aged 12 and 20.

9. The following table gives approximately the circumferences of circles corresponding to different radii :

C	15·7	20·1	31·4	44	52·2
r	2·5	3·2	5	7	8·3

Plot the values on squared paper, and from the graph determine the diameter of a circle whose circumference is 12·1 inches and the circumference of a circle whose radius is 2·8 inches.

10. For a given temperature, C degrees on a Centigrade are equal to F degrees on a Fahrenheit thermometer. The following table gives a series of corresponding values of F and C :

C	−10	−5	0	5	10	15	25	40
F	14	23	32	41	50	59	77	104

Draw a graph to shew the Fahrenheit reading corresponding to a given Centigrade temperature, and find the Fahrenheit readings corresponding to 12·5° C and 31° C.

By observing the form of the graph find the algebraical relation between F and C.

11. If W is the weight in ounces required to stretch an elastic string till its length is l inches, plot the following values of W and l :

W	2·5	3·75	6·25	7·5	10	11·25
l	8·5	8·7	9·1	9·3	9·7	9·9

From the graph determine the unstretched length of the string, and the weight the string will support when its length is 1 foot.

12. In a certain machine P is the force in pounds required to raise a weight of W pounds. The following corresponding values of P and W were obtained experimentally :

P	2·8	3·7	4·8	5·5	6·5	7·3	8	9·5	10·4	11·75
W	20	25	31·7	35·6	45	52·4	57·5	65	71	82·5

Draw the graph connecting P and W, and read off the value of P when $W = 60$. Also determine the law of the machine, and find from it the weight which could be raised by a force of 31·7 lbs.

13. The keeper of a hotel finds that when he has G guests a day his total daily profit is P pounds. If the following numbers are averages obtained by comparison of many days' accounts determine a simple relation between P and G.

G	21	27	29	32	35
P	1·8	2	3·2	4·5	6·6

For what number of guests would he just have no profit?

14. A man wishes to place in his catalogue a list of a certain class of fishing rods varying from 9 ft. to 16 ft. in length. Four sizes have been made at prices given in the following table:

9 ft.	11 ft. 9 in.	14 ft. 4 in.	16 ft.
15*s.*	22*s.*	31*s.*	38*s.*

Draw a graph to exhibit prices for rods of intermediate lengths, and from it determine the probable prices for rods of 13 ft. and 15 ft. 8 in.

15. In the Clergy Mutual Assurance Society the premium (£P) to insure £100 at different ages is given approximately by the following table:

Age	20	22	25	30	35	40	45	50	55
P	1·8	1·9	2·0	2·3	2·7	3·1	3·6	4·4	5·5

Illustrate the same statistics graphically, and estimate to the nearest shilling the premiums for persons aged 34 and 43.

16. The connection between the areas of equilateral triangles and their bases (in corresponding square and linear units) is given by the following table:

Area	·43	1·73	3·90	6·93	10·82	15·59
Base	1	2	3	4	5	6

Illustrate these results graphically, and determine the area of an equilateral triangle on a base of 2·4 ft.

17. A manufacturer has priced a certain set of lathes: the largest sells at £176, and the smallest at £40. He wishes to increase his prices so that the largest will sell at £200 and the smallest at £50. By means of a graph find an algebraical relation between the new price (P) and the old price (Q), and find to the nearest pound the new prices of lathes originally priced at £150, at £125. 10s., and at £78.

18. The mean temperature on the first day of each month, on an average of 50 years, had the following values:

Jan. 1, 37°;	May 1, 50°;	Sept. 1, 59°;
Feb. 1, 38°;	June 1, 57°;	Oct. 1, 54°;
Mar. 1, 40°;	July 1, 62°;	Nov. 1, 46°;
April 1, 45°;	Aug. 1, 62°;	Dec. 1, 41°.

Represent these variations by means of a smooth curve.

[The difference of length of different months may be neglected.]

19. The price in pence of a standard Troy ounce of silver on January 1st in each of the ten years 1891–1900 was

45, 40, 36, 29, 30, 31, 28, 27, 27, 28.

Draw a smooth curve shewing its value approximately at any time during these ten years.

20. Work the following three exercises graphically as if in each case one alone were given, taking in each case the simplest supposition which the information permits:

(i) The total yearly expense in keeping a school of 100 boys is £2100; what is the expense for 175 boys?

(ii) The expense is £2100 for 100 boys, £3050 for 200 boys: what is it for 175 boys?

(iii) The expenses in three cases are known as follows:

£2100 for 100 boys, £2650 for 150 boys, £3050 for 200 boys.

What is the probable expense for 175 boys?

21. A manufacturer wishes to stock a certain article in many sizes: at present he has five sizes made at the prices given below:

Length in inches	20	27	33	45	54
Price in shillings	11	14·5	20	35	48·5

Draw a graph to shew suitable prices for intermediate sizes, and find what the prices should be when the lengths are 30 in. and 46 in.

22. The salary of a clerk is increased each year by a fixed sum. After 6 years' service his salary is raised to £128, and after 15 years to £200. Draw a graph from which his salary may be read off for any year, and determine from it (i) his initial salary, (ii) the salary he should receive for his 21st year.

23. By measuring time along OX (1 inch for 1 hour), and distance along OY (1 inch for 10 miles) shew that a line may be drawn from O through the points (1, 8) (2, 16), (3, 24), .. to indicate distance travelled towards Y in a specified time at 8 miles an hour.

A starts from London at noon at 8 miles an hour; two hours later *B* starts, riding at 10 miles an hour. Find graphically at what time and at what distance from London *B* overtakes *A*. At what times will *A* and *B* be 8 miles apart? If *C* rides after *B*, starting at 3 p.m. at 15 miles an hour, find from the graphs

(i) the distances between *A*, *B*, and *C* at 5 p.m.;

(ii) the time when *C* is 8 miles behind *B*.

24. With the same conditions as in Ex. 23, shew how to draw a line from Y to indicate distance travelled from Y towards O at 6 miles an hour.

If O and Y represent two towns 45 miles apart, and if *A* walks from Y to O at 6 miles an hour while *B* walks from O to Y at 4 miles an hour, both starting at noon, find graphically their time and place of meeting.

Also read off from the graphs

(i) the times when they are 15 miles apart;

(ii) *A*'s distance from Y at 6.15 p.m.

25. At 8 a.m. *A* starts from P to ride to Q which is 48 miles distant. At the same time *B* sets out from Q to meet *A*. If *A* rides at 8 miles an hour, and rests half an hour at the end of every hour, while *B* walks uniformly at 4 miles an hour, find graphically

(i) the time and place of meeting;

(ii) the distance between *A* and *B* at 11 a.m.;

(iii) at what time they are 14 miles apart.

22. We shall now give some graphs of functions of higher degree than the first.

EXAMPLE. *Draw the graph of* $y = x^2$.

This is one of the most useful and interesting graphs the student will meet with; it is, therefore, important to plot the curve carefully on a suitable scale.

Take 0·4 of an inch as unit for x, and 0·1 of an inch for y, then *positive* values of x and y may be tabulated as follows:

x	0	0·5	1	1·5	2	2·5	3	3·5	4	...
y	0	0·25	1	2·25	4	6·25	9	12·25	16	...

Now if we take the following *negative* values of x

$$-0{\cdot}5,\ -1,\ -1{\cdot}5,\ -2,\ -2{\cdot}5,\ -3,\ -3{\cdot}5,\ -4,\ \ldots$$

we shall obtain the same series of values for y as before.

If the points we have now determined are plotted and connected by a continuous line drawn freehand, we shall obtain the curve shewn in Fig. 14. This curve is called a **parabola**.

There are three facts to be specially noted in this example.

(i) Since from the equation we have $x = +\sqrt{y}$, it follows that for every value of the ordinate we have two values of the abscissa, *equal in magnitude and opposite in sign*. Hence the graph is symmetrical with respect to the axis of y; so that after plotting with care enough points to determine the form of the graph in the first quadrant, its form in the second quadrant can be inferred without actually plotting any points in this quadrant. At the same time, in this and similar cases beginners are recommended to plot a few points in each quadrant through which the graph passes.

(ii) We observe that all the plotted points lie above the axis of x. This is evident from the equation: for since x^2 must be positive for all values of x, every ordinate obtained from the equation $y = x^2$ must be positive.

In like manner the student may shew that the graph of $y = -x^2$ is a curve similar in every respect to that in Fig. 14, but lying entirely below the axis of x.

(iii) As the numerical value of x increases that of y increases very rapidly. Hence, as there is no limit to the values which may be selected for x, it follows that the curve extends upwards and outwards to an infinite distance in both the first and second quadrants.

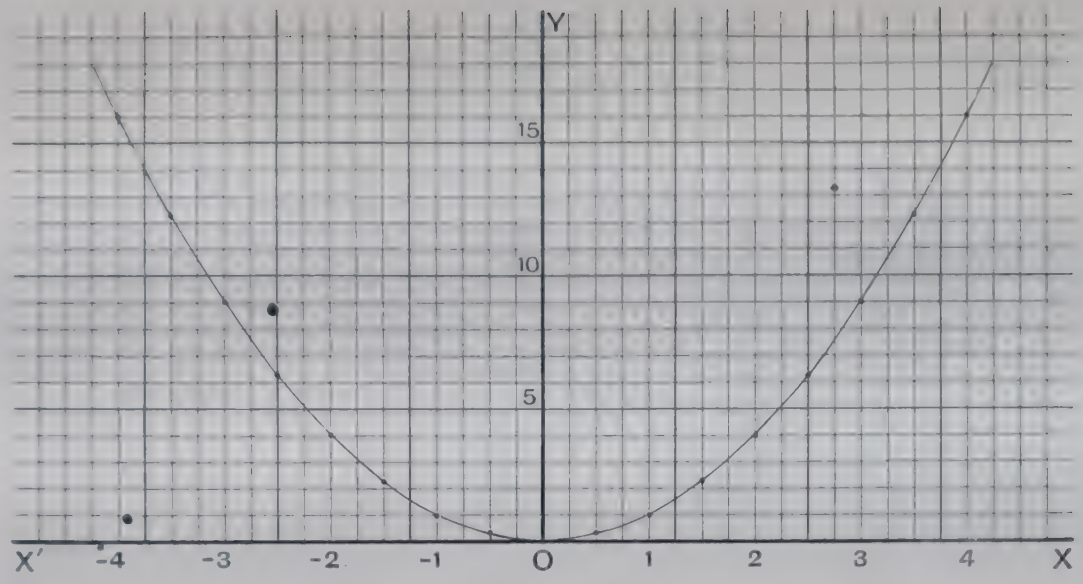

FIG. 14.

The student should draw this graph for himself with other units ; for example, half an inch may be taken as unit on each axis. Or 1 inch may be taken as unit for x, and 0·1 of an inch as unit for y.

23. Any equation of the form $y=ax^2$, where a is constant, will represent a parabola. If a is a positive integer, the curve will be as in Fig. 14 but will rise more steeply in the direction of OY. If a is a positive fraction, we shall have a flatter curve, extending more rapidly to right and left of OY. If a is negative, the curve will lie below the y-axis, and will be steeper or flatter than the graph of $y=x^2$, according as a is greater or less than unity. In every case the axis of x is a tangent to the curve at the origin.

24. We shall now discuss the graphs of some quadratic functions of the form ax^2+bx+c. It will be found that the curve is always a parabola, differing in shape and position according to the values of a, b, c.

EXAMPLE. *Find the graph of* $y=2x+\frac{x^2}{4}$.

Here the following arrangement will be found convenient :

x	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	−9
$2x$	6	4	2	0	−2	−4	−6	−8	−10	−12	−14	−16	−18
$\frac{x^2}{4}$	2·25	1	·25	0	·25	1	2·25	4	6·25	9	12·25	16	20·25
y	8·25	5	2·25	0	−1·75	−3	−3·75	−4	−3·75	−3	−1·75	0	2·25

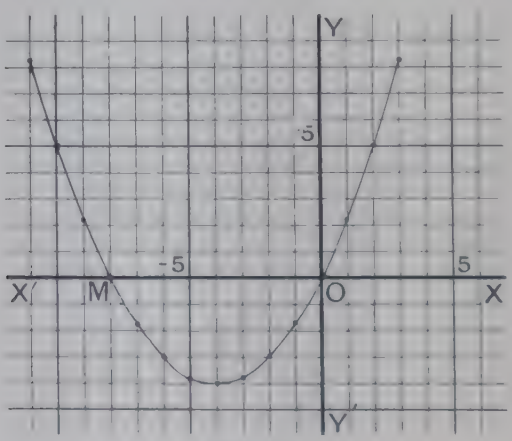

FIG. 15.

From the form of the equation it is evident that every positive value of x will yield a positive value of y, and that as x increases y

also increases. Hence the portion of the curve in the first quadrant lies as in Fig. 15, and can be extended indefinitely in this quadrant. In the present case only two or three positive values of x and y need be plotted, but more attention must be paid to the results arising out of negative values of x. It is found that the values of y are negative between $x=0$ and $x=-8$. When $x=-8$, $y=0$, and the curve crosses the x-axis; after this the values of y are positive.

25. In the last Example, since the value of $\frac{x^2}{4}+2x$ is represented by y, the *expression* $\frac{x^2}{4}+2x$ has a zero value when the ordinate is zero. Thus we can obtain the roots of the *equation* $\frac{x^2}{4}+2x=0$ by reading off the values of x at the points where the curve cuts the x-axis. These are $x=0$, $x=-8$, at the points O and M.

We can apply this method to an equation of any degree: thus if any function of x is represented by $f(x)$, a solution of the equation $f(x)=0$ may be obtained by plotting the graph of $y=f(x)$, and then measuring the intercepts made on the axis of x. These intercepts are values of x which make y equal to zero, and are therefore roots of $f(x)=0$.

26. In the graph of $y=x^2$ (Fig. 14) it will be noticed that as we pass from right to left along the curve the ordinate is constantly decreasing until it becomes zero at O; after this the ordinate begins to increase. The point at which this change takes place in a graph is known as a **turning point.** Thus the origin is a turning point of $y=x^2$, and of all curves represented by an equation of the form $y=ax^2$. Again in Fig. 15 there is a turning point at the point $(-4, -4)$. In each of these cases the algebraically least value of the ordinate is found at the turning point.

27. If a function gradually increases till it reaches a value a, which is algebraically greater than neighbouring values on either side, a is said to be a **maximum value** of the function.

If a function gradually decreases till it reaches a value b, which is algebraically less than neighbouring values on either side, b is said to be a **minimum** value of the function.

Let the function be represented by $f(x)$, then when $y=f(x)$ is treated graphically, it is evident that maximum and minimum values of $f(x)$ occur at the turning points, where the ordinates are algebraically greatest and least in the immediate vicinity of such points.

EXAMPLE. *Draw the graph of* $y = 3 - 4x - 4x^2$. *Thence find the roots of the equation* $4x^2 + 4x - 3 = 0$. *Shew that the expression* $3 - 4x - 4x^2$ *is positive for all real values of* x *between* 0·5 *and* − 1·5, *and negative for all real values of* x *outside these limits. Also find the maximum value of* $3 - 4x - 4x^2$.

Take the unit for x four times as great as that for y, and use the following table of values :

x	2	1·5	1	0·5	0	−0·5	−1	−1·5	−2	−2·5
$-4x$	−8	−6	−4	−2	0	2	4	6	8	10
$-4x^2$	−16	−9	−4	−1	0	−1	−4	−9	−16	−25
y	−21	−12	−5	0	3	4	3	0	−5	−12

After plotting these points we have the graph given in Fig. 16.

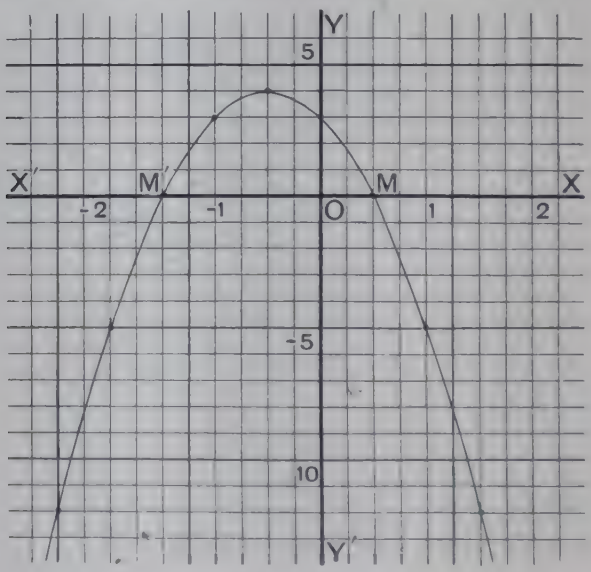

FIG. 16.

The roots of the equation $4x^2 + 4x - 3 = 0$ are the values of x which make y equal to 0. These are found at the points M and M′ where the curve cuts the x-axis. Thus the required roots are 0·5 and − 1·5.

Again between the points M and M′ the graph lies above the x-axis ; that is, the value of y, or $3 - 4x - 4x^2$, is positive so long as x lies between 0·5 and − 1·5, and is negative for other values of x.

The maximum value of the expression $3 - 4x - 4x^2$ is the value of the greatest ordinate in the graph, namely 4. This may also be obtained algebraically as follows:

$$3 - 4x - 4x^2 = 3 + 1 - (1 + 4x + 4x^2) = 4 - (1 + 2x)^2.$$

Now $(1 + 2x)^2$ must be positive for all real values of x except $x = -\frac{1}{2}$, in which case it vanishes, and the value of the expression reduces to 4, which is the greatest value it can have.

Note. Another method of dealing with examples of this class will be found in Art. 32.

EXAMPLES VI.

1. Draw the graphs of (i) $y = x^2$, (ii) $y = 8x^2$.

In (i) take 0·4″ as unit for x, 0·2″ as unit for y.

In (ii) ..... 1″ ..... x, 0·1″ ..... y.

2. On the same scale as in Ex. 1. (ii) draw the graph of $y = 16x^2$. Shew that it may also be simply deduced from the graph of Ex. 1. (ii).

3. Plot the graph of $y = x^2$, taking 1 inch as unit on both axes, and using the following values of x.

$$-0{\cdot}4,\ -0{\cdot}3,\ -0{\cdot}2,\ -0{\cdot}1,\ 0,\ 0{\cdot}1,\ 0{\cdot}2,\ 0{\cdot}3,\ 0{\cdot}4.$$

4. Draw the graphs of $y = x^2$, and $x = y^2$, and shew that they have only one common chord. Find its equation.

5. From the graphs, and also by calculation, shew that $y = \frac{x^2}{8}$ cuts $x = -y^2$ in only two points, and find their coordinates.

6. Draw the graphs of

(i) $y^2 = -4x$; (ii) $y = 2x - \frac{x^2}{4}$; (iii) $y = \frac{x^2}{4} + x - 2$.

In each case give the coordinates of the turning points.

7. Draw the graph of $y = x + x^2$. Shew also that it may be deduced from that of $y = x^2$.

8. Shew (i) graphically, (ii) algebraically, that the line $y = 2x - 3$ meets the curve $y = \frac{x^2}{4} + x - 2$ in one point only. Find its coordinates.

9. Find graphically the roots of the following equations to 2 places of decimals:

(i) $\frac{x^2}{4} + x - 2 = 0$; (ii) $x^2 - 2x = 4$; (iii) $4x^2 - 16x + 9 = 0$.

From the graphs deduce solutions of

(iv) $\frac{x^2}{4} + x - 2 = 6$; (v) $x^2 - 2x = 8$; (vi) $4x^2 - 16x + 9 = 6$.

10. On a large scale draw the graph of $x^2-7x+11$; hence find the roots of the equation $x^2-7x+11=0$, and the minimum value of the expression $x^2-7x+11$.

11. Find the minimum value of x^2-2x-4, and the maximum value of $5+4x-2x^2$.

12. Draw the graph of $y=(x-1)(x-2)$ and find the minimum value of $(x-1)(x-2)$. Measure, as accurately as you can, the values of x for which $(x-1)(x-2)$ is equal to 5 and 9 respectively. Verify algebraically.

13. Shew graphically that the expression x^2-2x-8 is negative for all values of x between -2 and 4, and positive of all values of x outside these limits.

28. The distance from the origin of any point $\mathsf{P}(x, y)$ is given by the relation $\mathsf{OP}^2=x^2+y^2$. Hence any equation of the form $x^2+y^2=a^2$, where a is constant, represents a circle, of radius a, whose centre is at the origin, since every point (x, y) which satisfies the equation is at a constant distance a from the origin.

EXAMPLE. *Solve graphically the simultaneous equations*

(i) $x^2+y^2=41$, (ii) $y=2x-3$.

The graph of (i) is a circle. Since the equation is satisfied by $x=4$, $y=5$ (the point P), the graph may be drawn by describing a circle with centre O and radius OP.

The graph of (ii) is a straight line, which cuts the axes at the points (1·5, 0), (0, −3).

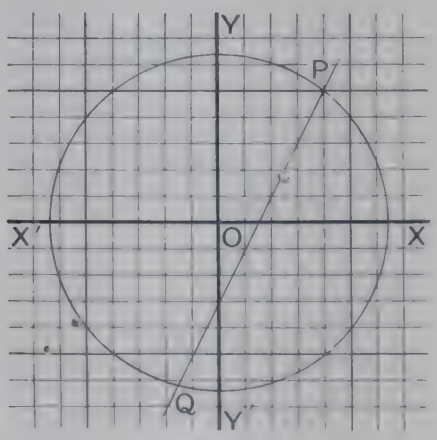

FIG. 17.

This line produced cuts the circle at P and Q. The coordinates of these points are (4, 5) and (−1·6, −6·2); thus the solution of the equations is given by

$x=4$, $y=5$, and $x=-1{\cdot}6$, $y=-6{\cdot}2$.

Infinite and Zero Values.

29. Consider the fraction $\frac{a}{x}$ in which the numerator a has a *certain fixed value*, and the denominator is a *quantity subject to change*; then it is clear that the smaller x becomes the larger does the value of the fraction $\frac{a}{x}$ become.

For instance

$$\frac{a}{\frac{1}{10}} = 10a, \qquad \frac{a}{\frac{1}{1000}} = 1000a, \qquad \frac{a}{\frac{1}{1000000}} = 1000000a.$$

By making the denominator x sufficiently small the value of the fraction $\frac{a}{x}$ can be made as large as we please; that is, if x is made *less than any quantity that can be named*, the value of $\frac{a}{x}$ will become *greater than any quantity that can be named.*

A quantity less than any assignable quantity is called **zero** and is denoted by the symbol 0.

A quantity greater than any assignable quantity is called **infinity** and is denoted by the symbol ∞.

We may now say briefly

$$\textit{when } x = 0, \textit{ the value of } \frac{a}{x} \textit{ is } \infty.$$

Again if x is a quantity which gradually increases and finally becomes *greater than any assignable quantity* the fraction becomes *smaller than any assignable quantity.* Or more briefly

$$\textit{when } x = \infty, \textit{ the value of } \frac{a}{x} \textit{ is } 0.$$

It should be observed that when the symbols for zero and infinity are used in the sense above explained, they are subject to the rules of signs which affect other algebraical symbols. Thus we shall find it convenient to use a concise statement such as "when $x = +0$, $y = +\infty$" to indicate that when a *very small and positive* value is given to x, the corresponding value of y is *very large and positive.*

EXAMPLE. *Find the graph of* $xy=4$. *Shew that it consists of two infinite branches, one in the first and the other in the third quadrant.*

The equation may be written in the form $y=\frac{4}{x}$,

from which it appears that when $x=0$, $y=\infty$ and when $x=\infty$, $y=0$. Also y is positive when x is positive, and negative when x is negative. Hence the graph must lie entirely in the first and third quadrants.

Take the positive and negative values of the variables separately.

(1) *Positive values:*

x	0	1	2	3	4	5	6	...	∞
y	∞	4	2	$1\frac{1}{3}$	1	·8	$\frac{2}{3}$	...	0

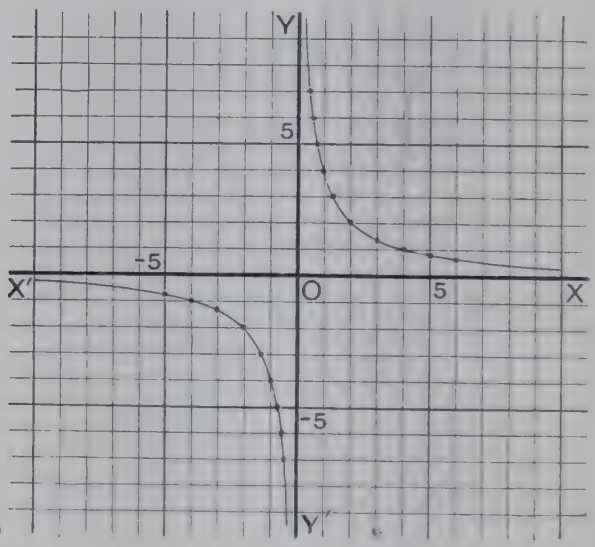

FIG. 18.

Graphically these values shew that as we recede further and further from the origin on the x-axis in the positive direction, the values of y are positive and become smaller and smaller. That is the graph is continually approaching the x-axis in such a way that by taking a sufficiently great positive value of x we obtain a point on the graph as near as we please to the x-axis but never actually reaching it until $x=\infty$. Similarly, as x becomes smaller and smaller the graph approaches more and more nearly to the positive end of the y-axis, never actually reaching it as long as x has any finite positive value, however small.

(2) *Negative values:*

x	-0	-1	-2	-3	-4	-5	...	$-\infty$
y	$-\infty$	-4	-2	$-1\frac{1}{3}$	-1	$-\frac{8}{5}$	...	-0

The portion of the graph obtained from these values is in the third quadrant as shewn in Fig. 18, and exactly similar to the portion already traced in the first quadrant. It should be noticed that as x passes from $+0$ to -0 the value of y changes from $+\infty$ to $-\infty$. Thus the graph which in the first quadrant has run away to an infinite distance on the positive side of the y-axis, reappears in the third quadrant coming from an infinite distance on the negative side of that axis. Similar remarks apply to the graph in its relation to the x-axis.

This curve is known as a **rectangular hyperbola**. Any equation of the form $xy = c$, where c is constant, will give a graph similar in form to that in Fig. 18.

30. When a curve continually approaches more and more nearly to a line without actually meeting it until an infinite distance is reached, such a line is said to be an **asymptote** to the curve. In the above case each of the axes is an asymptote.

31. In the simpler cases of graphs, sufficient accuracy can usually be obtained by plotting a few points, and there is little difficulty in selecting points with suitable coordinates. But in other cases, and especially when the graph has infinite branches, more care is needed. The most important thing to observe are (1) the values for which the function $f(x)$ becomes zero or infinite: and (2) the values which the function assumes for zero and infinite values of x. In other words, we determine the *general character* of the curve in the neighbourhood of the origin, the axes, and infinity. Greater accuracy of detail can then be secured by plotting points at discretion. The selection of such points will usually be suggested by the earlier stages of our work.

The existence of symmetry about either of the axes should also be noted. When an equation contains no *odd* powers of x, the graph is symmetrical with regard to the axis of y. Similarly the absence of odd powers of y indicates symmetry about the axis of x. [Compare Example in Art. 22.]

EXAMPLE. *Solve the following pairs of equations graphically:*

$$\text{(i)}\ \left.\begin{aligned} x-y&=2 \\ xy&=35 \end{aligned}\right\}; \qquad \text{(ii)}\ \left.\begin{aligned} x^2+y^2&=74 \\ xy&=35 \end{aligned}\right\}.$$

In each case we shall require the graph of $xy=35$. Proceeding as in the example of Art. 29, we find that the curve is a rectangular hyperbola lying in the first and third quadrants.

In (i) $x-y=2$ is a straight line QS making intercepts 2 and -2 on the axes.

In (ii) $x^2+y^2=74$ is a circle. Since the equation is satisfied by $x=5$, $y=7$, the graph can be drawn by finding this point (P), and describing a circle with centre O and radius OP.

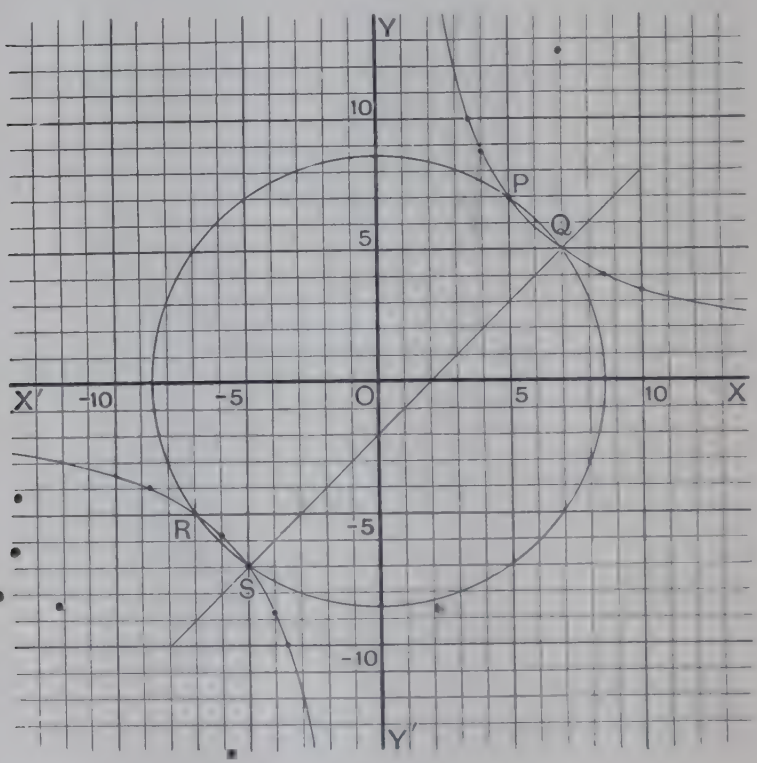

FIG. 19.

The roots of (i) are the coordinates of Q and S; that is,

$$x=7,\ y=5;\ \text{or}\ x=-5,\ y=-7.$$

The roots of (ii) are the coordinates of P, Q, R, and S: that is,

$$x=5,\ y=7;\ x=7,\ y=5;\quad x=-7,\ x=-5;\quad x=-5,\ y=-7.$$

32. Combination of two graphs. The method employed in the example on page 36 is quite general, and may be applied to functions of the third or higher degree, but the same results may often be more readily obtained by combining two graphs in the manner illustrated below.

EXAMPLE. *Solve the equation* $2x^2 - x - 3 = 0$ *graphically. Between what values of* x *is the expression* $2x^2 - x - 3$ *positive?*

Write the equation in the form $x^2 = \frac{x}{2} + \frac{3}{2}$.

Put $y_1 = x^2$.....(i), and $y_2 = \frac{x}{2} + \frac{3}{2}$.....(ii).

and plot the graphs of these equations, taking the x unit twice as great as the y unit.

For (i) we may use the values:

x	0	$\pm 0{\cdot}5$	± 1	$\pm 1{\cdot}5$	± 2
y	0	0·25	1	2·25	4

Thus we obtain the parabola POQ.

The intercepts of (ii) on the axes are -3, 1·5; thus the graph of (ii) is the straight line PQ.

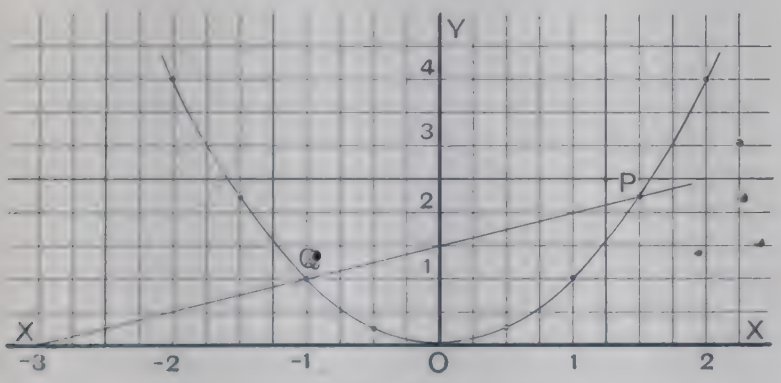

FIG. 20.

At the points of intersection, P and Q, the ordinates of (i) and (ii) are equal, that is $y_1 = y_2$; and the values of x at these points are 1·5 and -1. Hence for these values of x we have

$$x^2 = \frac{x}{2} + \frac{3}{2}, \quad \text{or} \quad 2x^2 - x - 3 = 0.$$

Thus the roots of the equation $2x^2 - x - 3 = 0$ are furnished by the abscissæ of the common points of the graphs of $y = x^2$ and $y = \frac{x}{2} + \frac{3}{2}$.

Again, the expression $2x^2 - x - 3$ is positive or negative according as y_1 is greater or less than y_2. From the graph we see y_1 is less than y_2 between Q and P, that is between $x = -1$ and $1{\cdot}5$, and y_1 is greater than y_2 for all other values of x. Hence $2x^2 - x - 3$ is positive for all values of x except such as lie between -1 and $1{\cdot}5$.

33. The solution of the last example might have been effected equally well by drawing the graphs of $y = 2x^2$ and $y = x + 3$. But if a number of quadratic equations have to be solved graphically it is convenient to reduce them to the form $x^2 = px + q$ as a first step. The graph of $y = x^2$ can then be plotted once for all on a suitable scale, and the line $y = px + q$ can be readily drawn for different values of p and q.

Equations of higher degree may be treated by a similar method. For example, the solution of such equations as

$$x^3 = px + q, \quad \text{or} \quad x^3 = ax^2 + bx + c$$

can be made to depend on the intersections of $y = x^3$ with the straight line $y = px + q$, or with the parabola $y = ax^2 + bx + c$.

EXAMPLE. *Find the real roots of the equations*

(i) $x^3 - 2{\cdot}5x - 3 = 0$; (ii) $x^3 - 3x + 2 = 0$.

Here we have to find the points of intersection of

(i) $y = x^3$, $y = 2{\cdot}5x + 3$; (ii) $y = x^3$, $y = 3x - 2$.

Plot the graphs of these equations, choosing the unit for x five times as great as that for y.

For $y = x^3$ the following values may be used :

x	0	$\pm 0{\cdot}5$	± 1	$\pm 1{\cdot}5$	± 2
y	0	$\pm 0{\cdot}125$	± 1	$\pm 3{\cdot}375$	± 8

The graph is shewn in Fig. 21. It may be noticed that the curve touches the x-axis at O, it crosses the axis at this point, and has symmetry in opposite quadrants.

The line $y = 2{\cdot}5x + 3$ joins the points (0, 3) and (2, 8), and meets $y = x^3$ only at the point P whose abscissa is 2. Thus 2 is the only real root of equation (i).

Again $y = 3x - 2$ joins the points (1, 1) and (0, -2); it *touches* $y = x^3$ at Q where $x = 1$, and cuts it at R where $x = -2$. Corresponding to the former point equation (ii) has *two equal* roots. Thus the roots are 1, 1, -2.

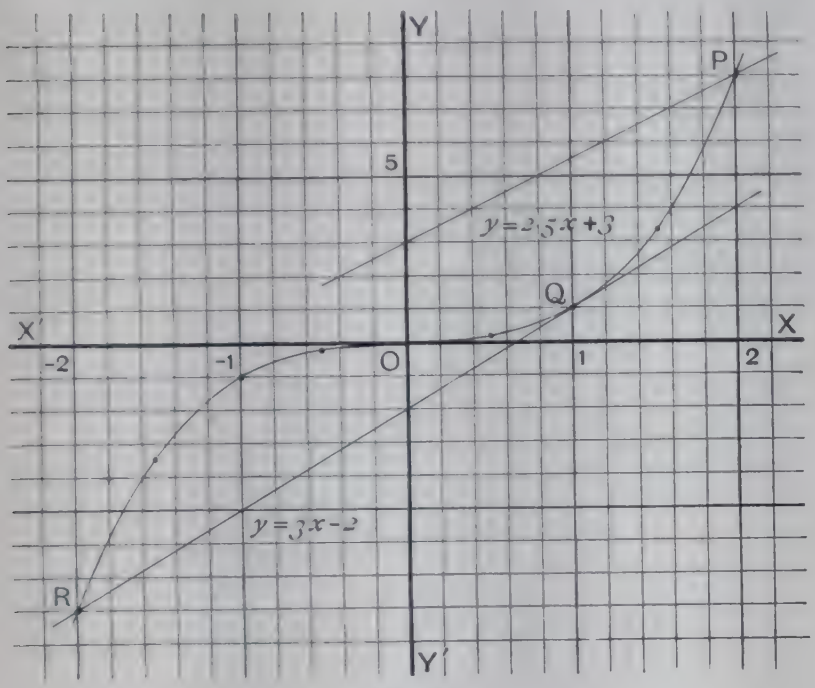

FIG. 21.

EXAMPLES VII.

1. Solve the following equations graphically.

(i) $x^2+y^2=53$, $y-x=5$;

(ii) $x^2+y^2=100$, $x+y=14$;

(iii) $x^2+y^2=34$, $2x+y=11$;

(iv) $x^2+y^2=36$, $4x+3y=12$.

[Approximate roots to be given to one place of decimals.]

2. Plot the graphs of $x^2+y^2=25$, $3x+4y=25$, and examine their relation to each other where they meet. Verify the result algebraically.

3. By the method of Art. 32 find graphically the roots of the following equations to two places of decimals:

(i) $\frac{x^2}{4}+x-2=0$; (ii) $x^2-2x=4$; (iii) $4x^2-16x+9=0$.

4. Solve graphically the equation $3+6x=x^2$, and find the maximum value of the expression $3+6x-x^2$.

5. Shew by the method of Art. 32 that the expression $4x^2+4x-3$ is negative of all real values of x between $0{\cdot}5$ and $-1{\cdot}5$, and positive for all real values of x outside those limits.

6. Draw the graphs of x^2 and of $3x+1$. By means of them find approximate values for the roots of $x^2-3x-1=0$.

7. Shew graphically that the expression x^2-4x+7 is positive for all real values of x.

8. On the same axes draw the graphs of

$$y=x^2, \quad y=x+6, \quad y=x-6, \quad y=-x+6, \quad y=-x-6.$$

Hence discuss the roots of the four equations

$$x^2-x-6=0, \quad x^2-x+6=0, \quad x^2+x-6=0, \quad x^2+x+6=0.$$

9. If x is real prove graphically that $5-4x-x^2$ is not greater than 9; and that $4x^2-4x+3$ is not less than 2. Between what values of x is the first expression positive?

10. The reciprocal of a number is multiplied by $2{\cdot}25$ and the product is added to the number. Find graphically what the number must be if the resulting expression has the least possible value.

11. Shew graphically that the expression $4x^2+2x-8{\cdot}75$ is positive for all real values of x except such as lie between $1{\cdot}25$ and $-1{\cdot}75$. For what value of x is the expression a minimum?

12. Solve the following pairs of equations graphically:

(i) $x+y=15$, $xy=36$; (ii) $x-y=3$, $xy=18$; (iii) $x^2+y^2=13$, $xy=6$.

13. From a graphical consideration of the following pairs of simultaneous equations:

(i) $x^2+y^2=a$, $xy=b$; (ii) $x+y=a$, $xy=b$;

explain why (i) has either *four* solutions or none, while (ii) has *two* solutions or none.

14. Draw the graph of $y=x^3$ between $x=-2$ and $x=2$, taking 1 inch as unit on each axis.

15. From the graphs of $y=x^3$ and $y=3x$ shew graphically that the roots of $x^3-3x=0$ are approximately $-1{\cdot}73$, 0, and $1{\cdot}73$.

[Take 1 inch as unit on each axis.]

16. Find graphically the real roots of the equations:

(i) $x^3+x-2=0$; (ii) $x^3-7x+6=0$.

17. Taking 1 inch as unit, plot the graph of $x^3 - 3x = 0$, taking the following values of x:

$$0,\ \pm\cdot2,\ \pm\cdot4,\ \pm\cdot6,\ \pm\cdot8,\ \pm1,\ \pm1\cdot2,\ \pm1\cdot4,\ \pm1\cdot6,\ \pm1\cdot8,\ \pm2.$$

Find the turning points, and the value of the maximum or minimum ordinates between the limits given. Verify the result of Ex. 15.

18. On the diagram of Ex. 17 draw the graph of $y + 2 = 0$, and thus verify the solution of the equation $x^3 - 3x + 2 = 0$ given in the Example on page 44.

19. On the same scale as on page 44, draw the graphs of $y = x^3$, and of $y = 2x^2 + x - 2$. Hence find the roots of the equation $x^3 - 2x^2 - x + 2 = 0$.

20. Taking the x unit ten times as great as the y unit plot the graphs of $x^3 - 4x^2$ and of $5x - 14$. Hence find the roots of the equation

$$x^3 - 4x^2 - 5x + 14 = 0$$

to three significant figures.

21. If p is a positive quantity, shew graphically that the equation $x^3 + px + q = 0$ can only have one real root.

Miscellaneous Applications of Graphs.

34. When two quantities x and y are so related that a change in one produces a proportional change in the other, their variations can always be expressed by an equation of the form $y = ax$, where a is constant. Hence in all such cases the graph which exhibits their variations is a straight line through the origin, and only one other point is required to determine the graph. For instance, such examples as deal with work and time, distance and time (when the speed is uniform), quantity and cost of material, principal and simple interest at a given rate per cent., may all be illustrated by linear graphs through the origin.

Note. It must be admitted that solutions of this kind are often very artificial, and they should be regarded mainly as exercises in ingenuity. The graphical treatment of a quite simple problem may often prove cumbrous and elaborate in detail, and in such a case a straightforward arithmetical or algebraical solution is to be preferred. For example, it is an unprofitable waste of time and skill to devise graphical solutions for certain types of easy problems which at most require only a few lines of very simple Arithmetic or Algebra. When, however, the answer to a question involves several allied results (as in Examples V. 23-25) a graphical method is often useful and interesting.

EXAMPLE 1. P *and* Q *are two towns* 30 *miles apart. At* 1 *p.m.* *X starts to walk from* Q *to* P *at* 3 *mi. an hour, and after walking two hours finds it necessary to run back for his watch. This he does at* $6\frac{2}{3}$ *mi. an hour, and after a delay of* 6 *minutes he again starts from* Q, *at* 4 *mi. an hour. Meanwhile Y starting from* P *at* 1 *p.m. sets out for* Q *at* 4 *mi. an hour; after walking for two hours, he spends half an hour with a friend from whom he borrows a bicycle on which he continues his journey at* 12 *mi. an hour. Draw graphs to shew the position of each man relative to* P *and* Q *at any time between* 1 *p.m. and* 5.30 *p.m. Also from the graphs find*

(i) *when and where X and Y meet;*

(ii) *at what times respectively they were* 18 *mi. and* 8 *mi. apart.*

In Fig. 22, on the opposite page, time is measured horizontally (1 inch to 1 hour), and distance vertically (1 inch to 10 miles). Thus each division on the horizontal axis represents 6 minutes and each division on the vertical axis stands for 1 mile.

The graph shewing the course of X is drawn downwards from Q; similarly Y's course is shewn by a graph drawn upwards from P.

At 3 p.m. X has gone 6 miles, therefore if A is taken 0·6 inch below the point which marks 3 p.m., QA is his graph for the first 2 hours.

To get back to Q at $6\frac{2}{3}$ miles an hour will take $6 \div 6\frac{2}{3}$, or $\frac{9}{10}$ of an hour. Hence B is the next point on his graph.

The delay at Q before he starts again at 4 miles an hour at 4 p.m. is represented by BC, which denotes 6 minutes. If D is taken 0·4 inch, representing 4 miles, vertically below 5 p.m., the line CDE completes the graph.

For Y's graph measure 0·8 inch vertically above 3 p.m. to F; then, since he walks 8 miles in 2 hours, PF is the first stage of the graph. The next half-hour is spent without advance towards Q; therefore the corresponding portion of the graph is FG.

GH represents the course of the bicycle ride at 12 miles an hour, and it will be found that it cuts X's graph at D.

Hence the point of meeting is at D, which is 4 miles from Q, and the time is 5 p.m. By inspection of the graphs we find LM and KH represent 18 and 8 miles respectively. The corresponding times are 3·3 and 4·5 hours; that is X and Y are 18 and 8 miles apart at 3.18 p.m. and 4.30 p.m. respectively.

They were also 18 mi. apart, approximately, at 18 min. before 3.

Note. The solution has here been given in full to illustrate and enforce the general principle on which the linear graphs depend. Solutions may usually be presented with less detail, and the results quickly obtained from a well-drawn diagram on a suitable scale.

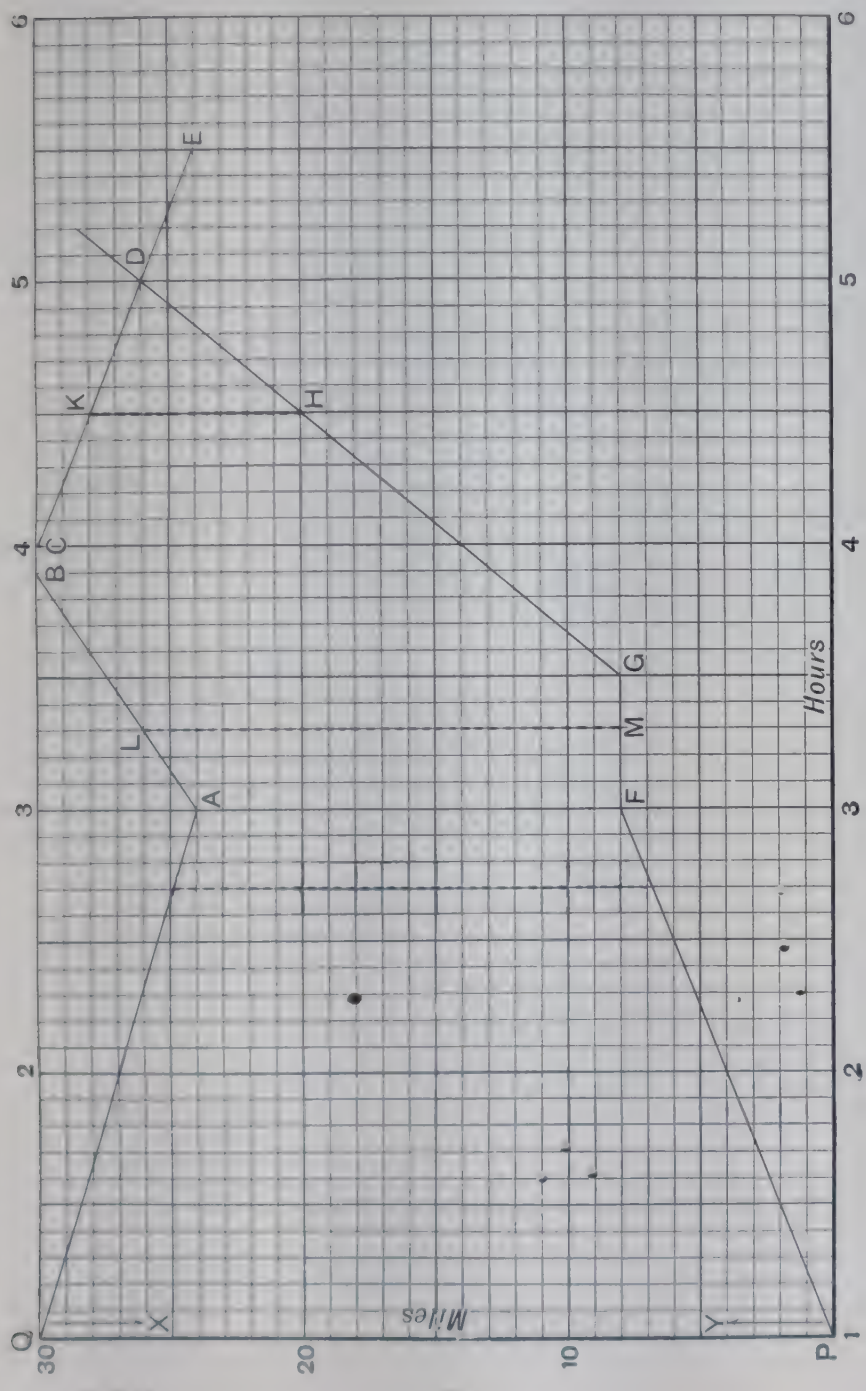

Fig. 22.

EXAMPLE 2. *A, B, and C run a race of* 300 *yards. A and C start from scratch, and A covers the distance in* 40 *seconds, beating C by* 60 *yards. B, with* 12 *yards' start, beats A by* 4 *seconds. Supposing the rates of running in each case to be uniform, find graphically the relative positions of the runners when B passes the winning post. Find also by how many yards B is ahead of A when the latter has run three-fourths of the course.*

In Fig. 23 let time be measured horizontally (0·5 inch to 10 seconds), and distance vertically (1 inch to 60 yards). O is the starting point for A and C; take OP equal to 0·2 inch, representing 12 yards, on the vertical axis; then P is B's starting point.

A's graph is drawn by joining O to the point which marks 40 seconds. From this point measure a vertical distance of 1 inch downwards to Q. Then since 1 inch represents 60 yards, Q is C's position when A is at the winning post, and OQ is C's graph.

Along the time-axis take 1·8 inch to R, representing 36 seconds; then PR is B's graph.

Through R draw a vertical line to meet the graphs of A and C in S and T respectively. Then S and T mark the positions of A and C when B passes the winning post.

By inspection RS and ST represent 30 and 54 yards respectively.

Thus B is 30 yards ahead of A, and A is 54 yards ahead of C.

Again, since A runs three-fourths of the course in 30 seconds, the difference of the corresponding ordinates of A's and B's graphs after 30 seconds will give the distance between A and B. By measurement we find VW = 0·45 inch, which represents 27 yards.

The student is recommended to draw a figure for himself on a scale twice as large as that given in Fig. 23.

35. When a variable quantity y is partly constant and partly proportional to a variable quantity x, the algebraical relation between x and y is of the form $y = ax + b$, where a and b are constant. The corresponding graph will therefore be a straight line; and since a straight line is completely determined when the positions of two points are known, it follows that, in all problems which can be illustrated by linear graphs, it is sufficient if the data furnish for each graph two independent pairs of simultaneous values of the variable quantities.

Some easy examples of this kind have already been given in Examples IV. See pages 18–20. We shall now work out two more examples.

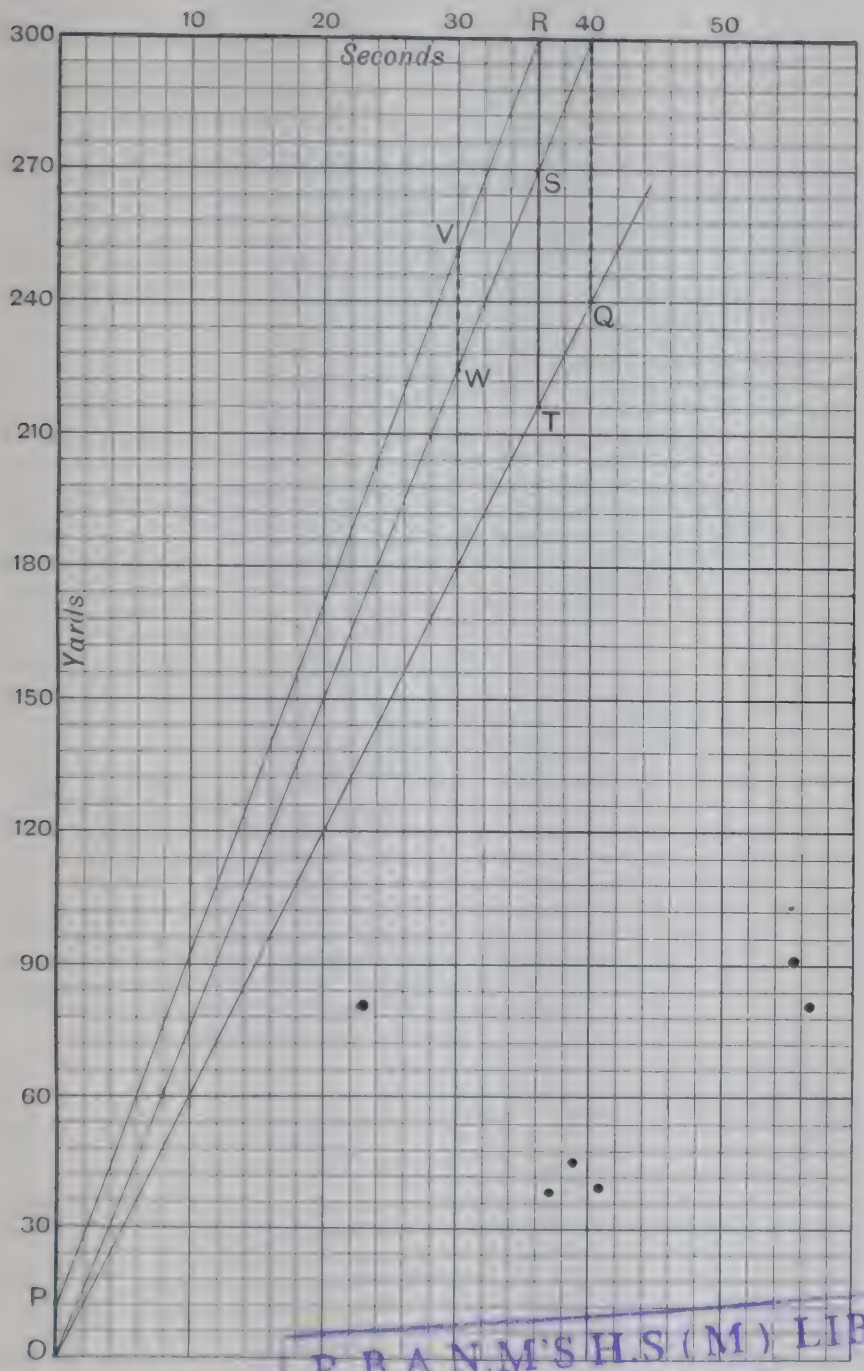

Fig. 23.

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EXAMPLE 1. *In a certain establishment the clerks are paid an initial salary for the first year, and this is annually increased by a fixed bonus, the initial salary and the bonus being different in different departments. A receives £130 in his 10th year, and £220 in his 19th. B, in another department, receives £140 in his 5th year and £180 in his 13th. Draw graphs to shew their salaries in different years. In what year do they receive equal salaries? Also find in what year A earns the same salary as that received by B for his 21st year.*

In Fig. 24 let each horizontal division represent 1 year; and let the salaries be measured vertically, beginning at 130, with 1 division to represent £2.

If the salary at the end of x years is denoted by £y, it is evident that in each case we have a relation of the form $y=ax+b$, where a and b are constant. Thus the variations of time and salary may be represented by linear graphs.

Since no bonus is received for the first year, $x=9$, when $y=130$, and $x=18$, when $y=220$. Thus the points P and Q are determined, and by joining them we have the graph for A's salary. Similarly the graph for B's salary is found by joining P′ (4, 140) and Q′ (12, 180).

These lines have the same ordinate and abscissa at L, where $x=16$, $y=200$. Thus A and B have the same salary when each have served 16 years, that is in their 17th year. Again B's salary at the end of 20 years is given by the ordinate of M, which is the same as that of Q which represents A's salary after 18 years.

Thus A's salary for his 19th year is equal to B's salary for his 21st year.

EXAMPLE 2. *Two sums of money are put out at simple interest at different rates per cent. In the first case the Amounts at the end of 6 years and 15 years are £260 and £350 respectively. In the second case the Amounts for 5 years and 20 years are £330 and £420. Draw graphs from which the Amounts may be read off for any year, and find the year in which the Principal with accrued Interest will amount to the same in the two cases. Also from the graphs read off the value of each Principal.*

When a sum of money is at simple interest for any number of years, we have

$$\text{Amount} = \text{Principal} + \text{Interest},$$

where 'Principal' is constant, and 'Interest' varies with the number of years. Hence the variations of Amount and Time may be represented by a linear graph in which x is taken to denote the number of years, and y the number of pounds in the corresponding Amount.

Here as the diagram is inconveniently large we shall merely indicate the steps of the solution which is similar in detail to that of the last example. The student should draw his own diagram.

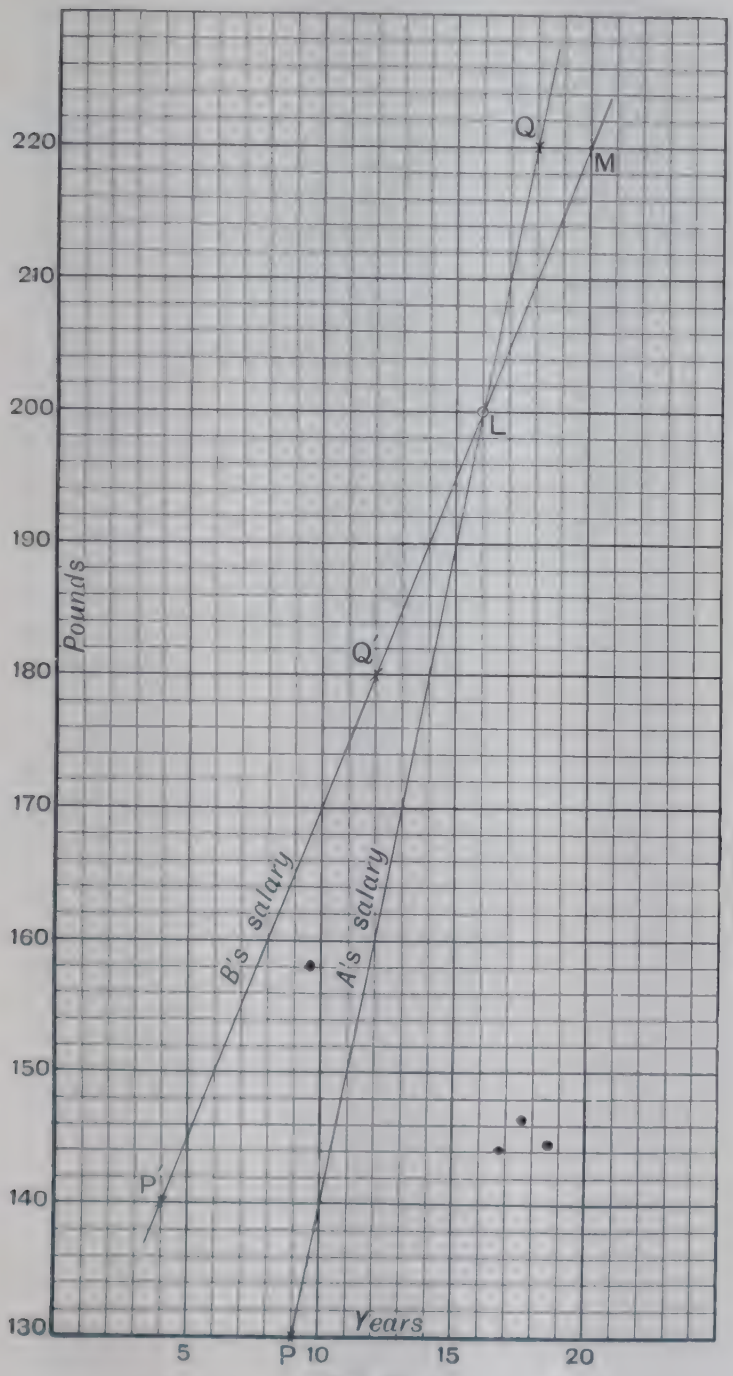
220
210
200
190
180
170
160
150
140
130
Pounds
Q
M
L
Q'
B's salary
A's salary
P'
Years
5
P
10
15
20

FIG. 24.

Measure time horizontally (1 inch to 10 years), and Amount vertically (1 inch to £40) beginning at £260.

The first graph is the line joining L (6, 260) and M (15, 350). The second graph is the line joining L′ (5, 330) and M′ (20, 420). In each of these lines the ordinate of any point gives the Amount for the number of years given by the corresponding abscissa.

Again LM, L′M′ intersect at a point P where $x=25$, $y=450$. Thus each Principal with its Interest amounts to £450 in 25 years.

When $x=0$ there is no Interest; thus the Principals will be obtained by reading off the values of the intercepts made by the two graphs on the y-axis. These are £200 and £300 respectively.

Note. To obtain the result $y=200$ it will be necessary to continue the y-axis downwards sufficiently far to shew this ordinate.

36. Some of the ordinary processes of Arithmetic lend themselves readily to graphical illustration. For example, the graph of $y=x^2$ may be used to furnish numerical square roots. For since $x=\sqrt{y}$, if a series of numbers are represented by ordinates, the corresponding abscissæ will give the square roots of those numbers. Similarly cube roots may be found from the graph of $y=x^3$.

EXAMPLE. *Draw a graph to find the cube roots of* 10 *and* 14 *correct to* 3 *places of decimals.*

The cube root of 10 is a little greater than 2; hence it will be sufficient to plot the graph of $y=x^3$, taking $x=2{\cdot}1$, $2{\cdot}2$, $2{\cdot}3$, $2{\cdot}4$, ... The corresponding ordinates are $9{\cdot}26$, $10{\cdot}65$, $12{\cdot}17$, $13{\cdot}82$, ... approximately.

When $x=2$, $y=8$. Take the axes through this point, and let the units for x and y be 10 inches and $0{\cdot}5$ inch respectively. The requisite portion of the curve is shewn in Fig. 25.

When $y=10$, by measurement we find $x=2{\cdot}154$.

Thus the cube root of $10=2{\cdot}154$.

When $y=14$, by measurement $x=2{\cdot}410$.

Thus the cube root of $14=2{\cdot}410$.

The graph may be used to read off the cube roots of all numbers between 8 and 14. For example, the cube roots of $8{\cdot}6$ and 13 are found to be $2{\cdot}050$ and $2{\cdot}350$.

Note. Solutions of this kind can only be regarded as a further *illustration* of the graphical method. As a *substitute* for arithmetical evolution they serve no useful purpose.

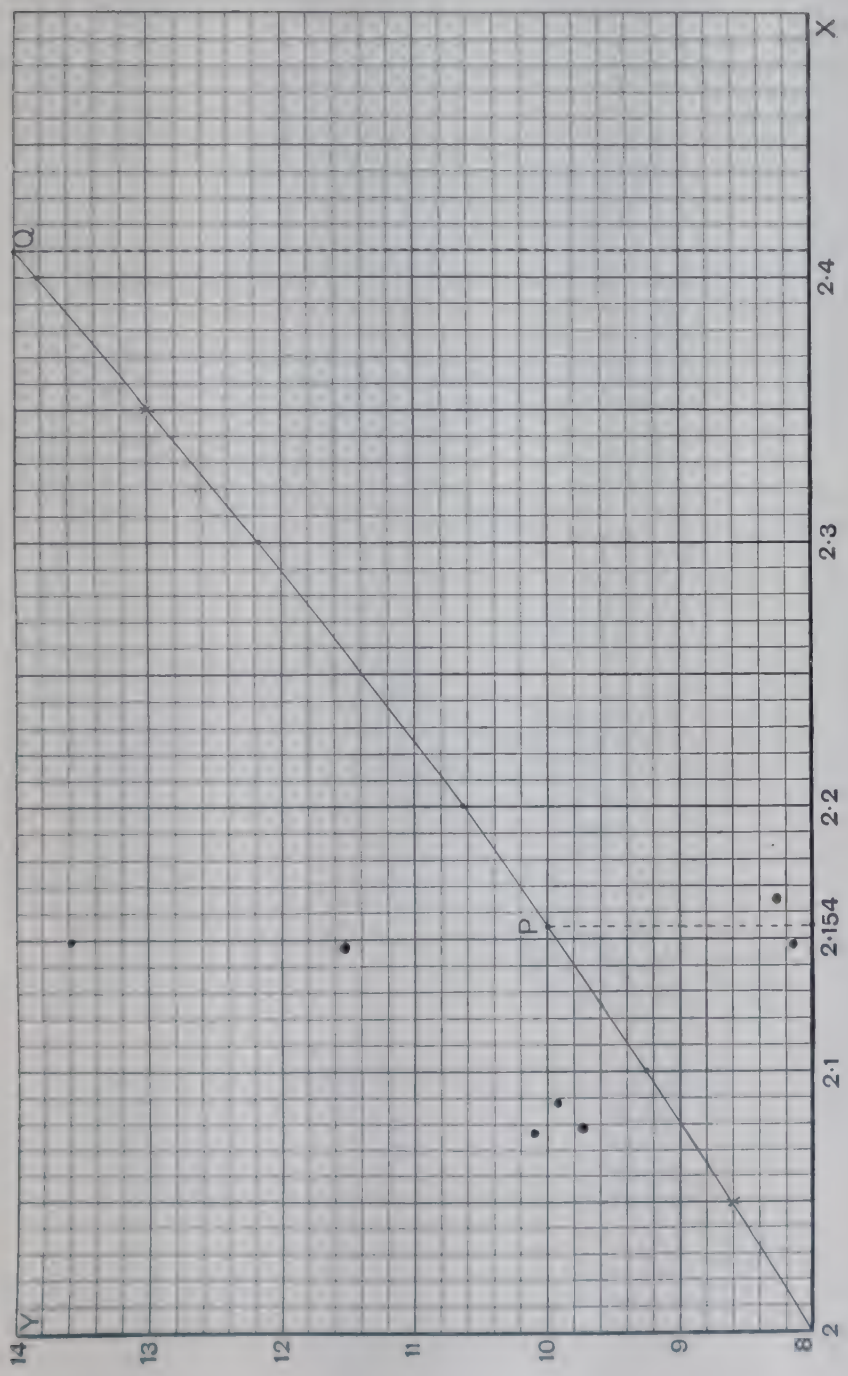

Fig. 25.

EXAMPLES VIII.

1. At noon A starts to walk at 6 miles an hour, and at 1.30 p.m. B follows on horseback at 8 miles an hour. When will B overtake A? Also find

(i) when A is 5 miles ahead of B;

(ii) when A is 3 miles behind B.

[Take 1 inch horizontally to represent 1 hour, and 1 inch vertically to represent 10 miles.]

2. By measuring time along OX (1 inch for 1 hour) and distance along OY (1 inch for 10 miles) shew how to draw lines

(i) from O to indicate distance travelled towards Y at 12 miles an hour;

(ii) from Y to indicate distance travelled towards O at 9 miles an hour.

If these are the rates of two men who ride towards each other from two places 60 miles apart, find from the graphs when they are first 18 miles from each other. Also find (to the nearest minute) their time of meeting.

3. Two bicyclists ride to meet each other from two places 95 miles apart. A starts at 8 a.m. at 10 miles an hour, and B starts at 9.30 a.m. at 15 miles an hour. Find graphically when and where they meet, and at what times they are $37\frac{1}{2}$ miles apart.

4. A and B start at the same time from London to Blisworth. A walking 4 miles an hour, B riding 9 miles an hour. B reaches Blisworth in 4 hours, and immediately rides back to London. After 2 hours' rest he starts again for Blisworth at the same rate. How far from London will he overtake A, who has in the meantime rested $6\frac{1}{4}$ hours?

5. At what distance from London, and at what time, will a train which leaves London for Rugby at 2.33 p.m., and goes at the rate of 35 miles an hour, meet a train which leaves Rugby at 1.45 p.m. and goes at the rate of 25 miles an hour, the distance between London and Rugby being 80 miles?

Also find at what times the trains are 24 miles apart, and how far apart they are at 4.9 p.m.

6. A, B, and C set out to walk from Bath to Bristol at 5, 6, and 4 miles an hour respectively. C starts 3 minutes before, and B 7 minutes after A. Draw graphs to shew (i) when and where A

overtakes C; (ii) when and where B overtakes A; (iii) C's position relative to the others after he has walked 45 minutes.

[Take 1 inch horizontally to represent 10 minutes, and 1 inch to the mile vertically.]

7. X and Y are two towns 35 miles apart. At 9 p.m. A starts to walk from X to Y at 4 miles an hour; after walking 8 miles he rests for half an hour and then completes his journey on horseback at 12 miles an hour. At 9.48 a.m. B starts to walk from Y to X at 3 miles and hour; find when and where A and B meet. Also find at what times they are $6\frac{1}{2}$ miles apart.

8. A can beat B by 20 yards in 120, and B can beat C by 10 yards in 50. Supposing their rates of running to be uniform, find graphically how much start A can give C in 120 yards so as to run a dead heat with him. If A, B, and C start together, where are A and C when B has run 80 yards?

9. A, B, and C run a race of 200 yards. A gives B a start of 8 yards, and C starts some seconds after A. A runs the distance in 25 seconds and beats C by 40 yards. B beats A by 1 second, and when he has been running 15 seconds, he is 48 yards ahead of C. Find graphically how many seconds C starts after A. Shew also from the graphs that if the three runners started level they would run a dead heat. [Take 1 inch to 40 yards, and 1 inch to 10 seconds.]

10. A cyclist has to ride 75 miles. He rides for a time at 9 miles an hour and then alters his speed to 15 miles an hour covering the distance in 7 hours. At what time did he change his speed?

11. A and B ride to meet each other from two towns X and Y which are 60 miles apart. A starts at 1 p.m., and B starts 36 minutes later. If they meet at 4 p.m., and A gets to Y at 6 p.m., find the time when B gets to X. Also find the times when they are 22 miles apart. When A is half-way between X and Y, where is B?

12. The distance from London to Bristol is 118 miles; if I were to set out at noon to cycle from London, riding 23 miles the first hour and decreasing my pace by 3 miles each successive hour, find graphically how long it would take me to reach Bristol. Also find approximately the time at which I should reach Faringdon, which is 48 miles from Bristol.

13. At 8 a.m. A begins a ride on a motor car at 20 miles an hour, and an hour and a half later B, starting from the same point, follows on his bicycle at 10 miles an hour. After riding 36 miles, A rests for 1 hr. 24 min., then rides back at 9 miles an hour. Find graphi-

cally when and where he meets B. Also find (i) at what time the riders were 21 miles apart, (ii) how far B will have ridden by the time A gets back to his starting point.

14. I row against a stream flowing $1\frac{1}{2}$ miles an hour to a certain point, and then turn back, stopping two miles short of the place whence I originally started. If the whole time occupied in rowing is 2 hrs. 10 mins. and my uniform speed in still water is $4\frac{1}{2}$ miles an hour, find graphically how far upstream I went.

[Take 1·2 of an inch horizontally to represent 1 hour, and 1 inch to 2 miles vertically.]

15. One train leaves Bristol at 3 p.m. and reaches London at 6 p.m.; a second train leaves London at 1.30 p.m. and arrives at Bristol at 6 p.m.; if both trains are supposed to travel uniformly, at what time will they meet? Shew from a graph that the time does not depend upon the distance between London and Bristol.

16. At 7.40 a.m. the ordinary train starts from Norwich and reaches London at 11.40 a.m.; the express starting from London at 9 a.m. arrives at Norwich at 11.40 a.m.: if both trains travel uniformly, find when they meet. Shew, as in Ex. 15, that the time is independent of the distance between London and Norwich, and verify this conclusion by solving an algebraical equation.

17. A boy starts from home and walks to school at the rate of 10 yards in 3 seconds, and is 20 seconds too soon. The next day he walks at the rate of 40 yards in 17 seconds, and is half a minute late. Find graphically the distance to the school, and shew that he would have been just in time if he had walked at the rate of 20 yards in 7 seconds.

18. The annual expenses of a Convalescent Home are partly constant and partly proportional to the number of inmates. The expenses were £384 for 12 patients and £432 for 16. Draw a graph to shew the expenses for any number of patients, and find from it the cost of maintaining 15.

In a rival establishment the expenses were £375 for 5, and £445 for 15 patients. Find graphically for what number of patients the cost would be the same in the two cases.

19. A body is moving in a straight line with varying velocity. The velocity at any instant is made up of the constant velocity with which it was projected (measured in feet per second) diminished by a retardation of a constant number of feet per second in every second. After 4 seconds the velocity was 320, and after 13 seconds it was 140. Draw a graph to shew the velocity at any time while the body is in motion.

A second body projected at the same time under similar conditions has a velocity of 450 after 5 seconds, and a velocity of 150 after 15 seconds. Shew graphically that they will both come to rest at the same time. Also find at what time the second body is moving 100 feet per second faster than the first, and determine from the graphs the velocity of projection in each case.

20. To provide for his two infant sons, a man left by his will two sums of money as separate investments at different rates of interest, on the condition that the principal sums with simple interest were to be paid over to his sons when the amounts were the same. After 5 years the first sum amounted to £451, and after 15 years to £533. After 10 years the second sum amounted to £432, and after 20 years to £544. Draw graphs from which the amounts may be read off for any year, and find after how many years the sons were entitled to receive their legacies.

Also determine from the graphs what the original sums were at the father's death.

21. Two vessels, A and B, are travelling at varying speeds. A's rate in any hour after the first is made up of a constant number of miles *increased* by a number of miles which is a constant fraction of the number of hours completed before that hour. B's rate in any hour after the first is made up of a constant number of miles *decreased* by a number of miles which is a constant fraction of the number of hours completed before that hour. In the 8^{th} and 24^{th} hours A travels 7 miles and 15 miles respectively. In the 5^{th} and 38^{th} hours B's rates are 23 miles and 12 miles. Draw graphs to shew the rates of each vessel for any hour, and find in what hour they are travelling at the same rate.

Find A's rate in the 38^{th} hour and B's in the 35^{th}. Shew also that in the 10^{th} hour B's speed is twice that of A.

22. In a certain examination the highest and lowest marks gained in a Latin paper were 153 and 51. These have to be reduced so that the maximum (120) is given to the first candidate, and the minimum (30) to the lowest. This is done by reducing all the marks in a certain ratio, and then increasing or diminishing them all by the same number. In a Greek paper the highest and lowest marks were 161 and 56; after a similar adjustment these become 100 and 40 respectively. Draw graphs from which all the reduced marks may be read off, and find the marks which should be finally given to a candidate who scored 102 in Latin and 126 in Greek.

Shew also that it is possible in one case for a candidate to receive equal marks in the two subjects both before and after reduction. What are the original and reduced marks in this case?

23. The table below shews the distances from London of certain stations, and the times of two trains, one up and one down. Supposing each run to be made at a constant speed, shew by a graph the distance of each train from London at any time, using 1 inch to represent 20 miles, and 3 inches to represent an hour.

Distance in miles.			↓	↑
	London,		4.30 p.m.	7.0 p.m.
$5\frac{1}{2}$	Willesden,	arrive	4.38 ...	
		depart	4.42 ...	(No intermediate
66	Northampton,	arrive	5.50 ...	stop.)
		depart	5.54 ...	
113	Birmingham,		7.0 ...	5.0 p.m.

At what point do they pass one another, and how far is each from London at 5.30? Which of the three runs by the stopping train is the fastest?

24. Taking 1 inch as unit for x, and 0·5 as unit for y, draw the graph of $y=x^2$, and employ it to find the squares of 0·72, 1·7, 3·4: and the square roots of 7·56, 5·29, 9·61.

25. Draw the graph of $y=\sqrt{x}$ taking the unit for y five times as great as that for x.

By means of this curve check the values of the square roots found in Ex. 24.

26. Draw a graph which will give the square roots of all numbers between 25 and 36, to three places of decimals.

[Take the origin at the point 5, 25, and use the same units as in the Example of Art. 36.]

27. From the graph of $y=x^3$ (on the scale of the diagram of Art 36) find the values of $\sqrt[3]{9}$ and $\sqrt[3]{9{\cdot}8}$ to 4 significant figures.

28. A boy who was ignorant of the rule for cube root required the value of $\sqrt[3]{14{\cdot}71}$. He plotted the graph of $y=x^3$, using for x the values 2·2, 2·3, 2·4, 2·5, and found 2·45 as the value of the cube root. Verify this process in detail. From the same graph find the value of $\sqrt[3]{13{\cdot}8}$ to two places of decimals.

29. Draw a graph which will give the cube roots of all numbers between 27 and 64 correct to 3 places of decimals.

Read off the cube roots of 44, 60; and the cubes of 3·42, 3·659.

ANSWERS.

I. Page 4.

7. 36. **8.** 32. **9.** 25. **11.** 1·2 sq. cm.

12. $y=3x$. Any point whose ordinate is equal to three times its abscissa.

14. (i) (8, 5); (ii) (10, 10).

15. (i) (4, 5); (ii) (4, 5); (iii) (−4, −5); (iv) (−4, −5).

16. (i) 17; (ii) 17; (iii) 2·5″; (iv) 2·5″.

17. (i) and (ii) 5; (iii) and (iv) 17; (v) and (vi) 37.

19. The lines are $x=5$, $y=8$. The point (5, 8).

20. 10. **23.** 68 units. **24.** 10, 13, 5, 5, 3.

25. A circle of radius 13 whose centre is at the origin.

27. 10, 12, 16, 6, 0. At the points (0, 10), (−5, 0).

II. Page 10.

21. 32 units of area. **22.** 1 sq. in.

23. 72 units of area. **24.** 0·64 sq. cm.

III. Page 15.

1. $x=1$, $y=5$. **2.** $x=2$, $y=10$. **3.** $x=3$, $y=12$.

4. $x=3$, $y=-2$. **5.** $x=4$, $y=2$. **6.** $x=6$, $y=8$.

7. $x=-2$, $y=4$. **8.** $x=0$, $y=-3$. **9.** $x=-3$, $y=0$.

10. $x=-4$, $y=5$. **11.** At the point (0, 21). **12.** $3x+4y=7$.

IV. Page 17.

2. $x=1$, $y=18$. 3. 9; 2·4.
4. 5 in each case. 18·5 units. 5. 2·5, 1·7.
6. (3, 4), (4, 1), (−3, 2). 7. 7·5 sq. in.
8. 2·60, 5·63, 4·16, 5·77. 10. (i) 53·7 grains; (ii) 0·2.
11. 39·3; 91·6; $y=0{\cdot}393x$. 12. 90; 72. 13. 5*s.*; 87.
14. 112; 168; 78. $y=\frac{10}{11}x-70$.
15. $y=100+\frac{x}{10}$; £350; 4250.

V. Page 26.

5. $11y=3x+35$. 16; 25. 6. $y=0{\cdot}21x+1{\cdot}37$.
7. $y=0{\cdot}4x+1{\cdot}6$. 9·2; 3. 8. 45·96; 39·40.
9. 3·85 in.; 17·6 in. 10. 54·5° F.; 86·9° F.; $F=32+\frac{9}{5}$ C.
11. 8·1 in.; 24·375 oz. 12. 8·6; $P=0{\cdot}14\,W+0{\cdot}2$; 225 lbs.
13. $P=0{\cdot}6G-14{\cdot}4$; 24. 14. 26*s.*; 36*s.* 6*d.*
15. £2. 12*s.*; £3. 8*s.* 16. 2·49 sq. ft.
17. $P=1{\cdot}1Q+6$; £171; £144; £92.
20. (i) 3675; (ii) £2812. 10*s.*; (iii) £2860.
21. 17*s.*; 26*s.* 6*d.* 22. £80; £240.
23. 6 p.m., 48 mi. from London. At 4 and 8 p.m.
(i) *B* 4 mi. behind *A*; *C* 6 mi. behind *B*. (ii) 4·21 p.m.
24. 4·30 p.m., 18 mi. from O. (i) At 3 and 6 p.m. (ii) 20 mi.
25. (i) 1 p.m., 28 mi. from P; (ii) 20 mi.; (iii) 11.30 a.m.

VI. Page 37.

4. $y=x$. 5. (0, 0), (−4, 2).
6. (i) (0, 0); (ii) (4, 4); (iii) (−2, −3). 8. (2, 1).
9. (i) 1·46, −5·46; (ii) 3·24, −1·24; (iii) 3·32, 0·68;
(iv) 4, −8; (v) 4, −2; (vi) 1·5, 2·5.
10. 2·38, 4·62; −1·25. 11. −5; 7.
12. −0·25; 3·79, −0·79; 4·62, −1·62.

VII. Page 45.

1. (i) $x=2$, or -7; $y=7$, or -2. (ii) $x=8$, or 6; $y=6$, or 8. (iii) $x=3$, or $-5{\cdot}8$; $y=5$, or $-0{\cdot}6$. (iv) $x=5{\cdot}2$, or $-1{\cdot}3$; $y=-2{\cdot}9$, or $5{\cdot}7$.
2. The straight line $3x+4y=25$ *touches* the circle $x^2+y^2=25$ at the point (3, 4).
3. (i) $1{\cdot}46$, $-5{\cdot}46$; (ii) $3{\cdot}24$, $-1{\cdot}24$; (iii) $3{\cdot}32$, $0{\cdot}68$.
4. $6{\cdot}46$, $-0{\cdot}46$. 12. 6. $3{\cdot}30$, $-0{\cdot}30$.
9. -5 and 1. 10. $1{\cdot}5$. 11. $-0{\cdot}25$.
12. (i) $x=12$, or 3; $y=3$, or 12; (ii) $x=6$, or -3; $y=-3$, or -6. (iii) $x=2, 3, -3, -2$; $y=3, 2, -2, -3$.
16. (i) 1; (ii) 1, 2, 3.
17. Maximum ordinates ($=2$) at the points $(-1, 2)$, $(1, -2)$.
19. -1, 1, 2. 20. $1{\cdot}59$, $4{\cdot}41$, -2.

VIII. Page 56.

1. 6 p.m.; (i) 3.30 p.m.; (ii) 7 p.m. 2. (i) 2 p.m.; 2·52 p.m.
3. 47 mi. from *A*'s starting place at 12.42 p.m.
11.12 a.m. and 2.12 p.m. 4. 27 mi.
5. 35 mi. from London at 3.33 p.m. 3.9 p.m.; 3.51 p.m.; 36 mi.
6. (i) 15 mi. after *C*'s start, 1 mi. from Bath;
(ii) 45 .... $3\frac{1}{2}$ mi. .....;
(iii) half a mile behind *A* and *B*.
7. 9 mi. from Y at 12.48 p.m. 12.18 p.m. and 1.18 p.m.
8. 40 yds. *A* 16 yds. ahead, *C* 16 yds. behind.
9. 5 secs. 10. 5 hours from the start.
11. 7.36 p.m.; 3 p.m. and 5 p.m.; 19 mi. from Y.
12. 7 hours. 4.47 p.m. 13. 12.12 p.m. (i) 11 a.m.; (ii) 57 mi.
14. 5 mi. 15. 4.12 p.m. 16. 10.4 a.m.
17. 400 yds. 18. £420. 20 for £480.

19. After 10 secs. 400 ft. per sec. ; 600 ft. per sec.

20. 30 years. £410 ; £320. **21.** 26th hour. 22 min.; 13 mi.

22. 75 in Latin ; 80 in Greek. 74 and 50.

23. At Northampton. 48·4 mi., 84·8 mi. The quickest run is from Willesden to Northampton.

24. 0·52, 2·9, 11·6 ; 2·75, 2·3, 3·1. **27.** 2·080, 2·140.

28. 2·40. **29.** 3·530, 3·915 ; 40, 49.

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